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A TREATISE

Edw. Brunell
ON
St. John's College
October 5. 1847.

THE MOTION OF A RIGID BODY.

BY

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S. W. 1837

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It has been the wish of the compiler of the following work to exhibit in a brief and compact form the leading principles of the Motion of a Rigid Body, under an impression that a syllabus of this kind is best adapted for the combination of oral teaching and reference to books which is the present practice of the University. While in this view it appears convenient to present the propositions of the subject apart from their applications, it will not be supposed that extensive practice in the solution of problems can be dispensed with by any who would understand the theory of this or any other portion of Mechanics. The author has not however thought it requisite to include in the present book any specimens of the use of its principles, from an expectation that Mr Walton's collection will be in the hands of every reader and supply him with guidance and exercise.

A few pages of examples have been added as supplementary to those already accessible. Most of them are aimed at particular difficulties and misconceptions to which readers have in experience been found liable, with a belief that questions of this kind are most valuable as a means of starting reflection, whereby the student is enabled to detect and rectify his misapprehensions.

St John's College,
October, 1847.

ERRATUM.

Page 61, line 7 from bottom *for* rotation *read* translation.

SECTION I.

GEOMETRICAL PROPERTIES OF A RIGID BODY.

1. DEF. A body of finite dimensions in which all points retain an invariable position with respect to one another is called a rigid body.

A rigid body is thus distinguished from bodies flexible, compressible, extensible and fluid; because in these, assigned points may have different relative positions while the continuity of the substance is preserved, whereas the particles of a rigid body may occupy different positions in space but still preserve the same relative positions among one another.

Without asserting that any substance in nature strictly fulfils this definition of a rigid body, mathematical calculations proceed upon this conception of it with a view to distinctness and simplicity. When the results of these calculations are employed in practice, they require small corrections, by reason of the substances to which they are applied not possessing completely the properties of the imaginary body on which we have reasoned. If however the forces in action are not adequate to produce an appreciable change of form, the body subject to them may be regarded during their action as perfectly rigid and incapable of any change in shape or consistence. Iron, for instance, will sensibly depart from its primitive form when a force sufficiently great is applied to it, and is found to be flexible, extensible, and compressible, but so long as the forces acting upon it are so small as to be incapable of altering its shape to a measurable extent, it may be regarded as rigid according to the definition.

2. The definition thus given of a rigid body is only one out of many instances where, in the application of Analysis to Physical Science, hypotheses are needful to simplify the pro-

blems which practice presents. Strings, for instance, are considered weightless and perfectly flexible, the surfaces of bodies smooth, planes accordant with their geometrical definition, none of which circumstances are ever perfectly realized. The results of calculation therefore on these or similar hypotheses are first approximations to the solution of the questions which really arise, requiring corrections to bring them to accuracy whenever the conditions on which they rest are appreciably violated. This manner of proceeding, which in any case would perhaps have the recommendation of simplicity, is unavoidable while Analysis is imperfect. Problems thus reduced and simplified may be within the reach of our instruments of calculation, when they would be beyond it if they retain all the complexity which their natural state presents.

3. A body of finite size may be conceived to consist of an infinite number of elementary portions each of inappreciable volume but finite mass, and exhibited in position by reference to co-ordinate axes, so that the place of any one of these elements may be represented by the co-ordinates of any point of it. Such elements are sometimes termed the particles of the body.

Obs. The axes of co-ordinates are to be considered rectangular when the contrary is not expressed.

4 *Moment of Inertia.*

Def. The moment of inertia of a rigid body about a given axis is the sum of the products of the mass of each particle of the body and the square of the distance of that particle from the axis.

Thus if m be the mass of a particle of a rigid body, r its distance from a fixed straight line, $\Sigma(mr^2)$ is the moment of inertia of the body about that line, the summation extending throughout the whole body. If ρ be the density at the point x, y, z of the body, $\int_x\int_y\int_z \rho r^2$ is the moment of inertia of the body with respect to the axis from which r is measured, the integration being performed throughout the body. The defi-

nition given above is to be regarded as a statement in words of this analytical expression, which has required nomenclature by the frequency of its occurrence in dynamical investigations.

The computation of the moment of inertia of a given body about a given axis thus amounts to the evaluation of a definite integral which is equivalent to the summation of elements, but it is often facilitated by some of the following properties.

5. To connect the moment of inertia of a rigid body about any axis with its moment of inertia about a parallel axis through the centre of gravity.

Let P be a particle whose mass is m (fig. 1), A, G the traces of two straight lines on a plane through P perpendicular to each, the latter of these straight lines being drawn through the centre of gravity of the rigid body of which P is a particle. Join PA, PG, and draw PN perpendicular to AG or AG produced.

$$\text{Then} \quad PA^2 = PG^2 + AG^2 + 2\,AG \,.\, GN,$$

$$\text{and} \quad \Sigma(m \,.\, GN) = 0,$$

$$\therefore \quad \Sigma(m \,.\, PA^2) = \Sigma(m \,.\, PG^2) + AG^2 . \Sigma(m),$$

or the moment of inertia of the body about the axis A is greater than that about the axis G by the product of the mass of the body and the square of the distance between the axes.

6. The moment of inertia of a body of mass M about an axis through its centre of gravity is usually denoted by Mk^2, k being dependent on the direction of the axis. The moment of inertia of the body about a parallel axis at distance h will then be $M(k^2 + h^2)$. The length $\sqrt{h^2 + k^2}$ is sometimes called the radius of gyration of the body about the axis considered, and k similarly the radius of gyration about a parallel axis through the centre of gravity.

7. If a circular cylinder be described so that the centre of gravity of the body is in its axis, the moment of inertia of

the body about all generating lines of this cylinder is the same.

8. If there be any two parallel axes A and B at distances a and b from the centre of gravity of the body whose mass is M,

moment of inertia about $A - Ma^2$ = moment about $B - Mb^2$.

Hence when the moment of inertia of a body about any axis is given, that about any other parallel axis can be deduced if the position of the centre of gravity be known.

9. DEF. A solid of finite size bounded by two parallel planes, the distance between which is infinitely less than the other dimensions of the body, is called a lamina.

The moment of inertia of a lamina about an axis perpendicular to its plane is the sum of its moments of inertia about any two perpendicular axes in its plane drawn from the point where the former axis meets the plane.

For if P (fig. 2) be a particle of the lamina of mass m, PN, PM perpendiculars on two straight lines AB, AC at right angles to one another and in the plane of the lamina, and if AP be joined

$$AP^2 = PM^2 + PN^2;$$

$$\therefore \quad \Sigma(m.AP^2) = \Sigma(m.PM^2) + \Sigma(m.PN^2),$$

i.e. the moments of inertia about AB and AC are together equal to the moment of inertia about an axis through A perpendicular to the plane of the lamina.

10. To find the moment of inertia of a rigid body about a given straight line through the origin of co-ordinates to which the body is referred.

Let x, y, z be the co-ordinates of a particle of mass m,

$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$ the equations to the given straight line,

where $\alpha^2 + \beta^2 + \gamma^2 = 1$.

The distance of the point x, y, z from this line

$$= \{x^2 + y^2 + z^2 - (\alpha x + \beta y + \gamma z)^2\}^{\frac{1}{2}}.$$

$$= \{\alpha^2(y^2+z^2)+\beta^2(x^2+z^2)+\gamma^2(x^2+y^2)-2\beta\gamma yz-2\alpha\gamma xz-2\alpha\beta xy\}^{\frac{1}{2}}.$$

Let $A = \Sigma m (y^2 + z^2)$
$B = \Sigma m (x^2 + z^2)$
$C = \Sigma m (x^2 + y^2)$
} be the moments of inertia of the body about the co-ordinate axes,

Q the moment of inertia of the body about the given axis.

$$\therefore Q = A\alpha^2 + B\beta^2 + C\gamma^2 - 2\beta\gamma\Sigma(myz) - 2\alpha\gamma\Sigma(mxz) - 2\alpha\beta\Sigma(mxy).$$

11. *Principal Axes.*

DEF. If x, y, z be co-ordinates of a particle of mass m belonging to a rigid body, and if $\Sigma(myz) = 0$, $\Sigma(mxz) = 0$, $\Sigma(mxy) = 0$, the summations extending throughout the whole body, the axes of co-ordinates are then called principal axes at the point which is adopted as origin.

OBS. Principal axes are not defined separately, but in a system. Wherefore if one of the preceding conditions be fulfilled it may not thence be concluded that one of the axes of co-ordinates is a principal axis.

12. Through any point in space there may be drawn at least one system of principal axes of a given body.

If Q be the moment of inertia of the body about the axis whose direction cosines are α, β, γ,

$$Q = A\alpha^2 + B\beta^2 + C\gamma^2 - 2\beta\gamma\Sigma(myz) - 2\alpha\gamma\Sigma(mxz) - 2\alpha\beta\Sigma(mxy).$$

Now this is the equation to a surface of the second order, where $Q^{-\frac{1}{2}}$ represents the radius vector in the direction expressed by α, β, γ. But one system at least of co-ordinate axes, the principal axes of the surface, can be adopted for which the equation to the surface will not involve the products $\beta\gamma$, $\alpha\gamma$, $\alpha\beta$; therefore with these co-ordinate axes $\Sigma(myz) = 0$, $\Sigma(mxz) = 0$, $\Sigma(mxy) = 0$, or the axes are a system of principal axes.

Since moments of inertia are necessarily positive quantities, the surface is an ellipsoid.

A demonstration of this proposition without reference to the properties of the ellipsoid will be found in the Cambridge Mathematical Journal, Vol. I.

13. If the ellipsoid become a figure of revolution, which will be the case if the expressions

$$A + \frac{\Sigma(mxy) \cdot \Sigma(mxz)}{\Sigma(myz)}, \quad B + \frac{\Sigma(mxy) \cdot \Sigma(myz)}{\Sigma(mxz)},$$

$$C + \frac{\Sigma(mxz) \cdot \Sigma(myz)}{\Sigma(mxy)},$$

be equal and finite, (Hymers' Geometry of three dimensions, 149) then while one principal axis is fixed in direction, any pair of rectangular axes perpendicular to it form with it a system of principal axes.

14. If the ellipsoid become a sphere, which will be the case if

$$A = B = C, \quad \Sigma(myz) = 0, \quad \Sigma(mxz) = 0, \quad \Sigma(mxz) = 0,$$

then any system of rectangular axes is a system of principal axes at the origin.

Hence when any point is adopted for origin, there is always at least one determinate system of principal axes belonging to it, and in certain cases an infinite number of such systems.

Obs. The origin of co-ordinates in this proposition is not necessarily a point in the body or in any way restricted.

15. If the axis of z be a principal axis, $(\Sigma mxz) = 0$, $\Sigma(myz) = 0$, for any system of rectangular co-ordinate axes.

For let $x'y'$ designate the co-ordinates of m referred to the principal axes;

$$\therefore \quad \Sigma(mx'z) = 0, \quad \Sigma(my'z) = 0.$$

Let x' and x make an angle θ,

$$\therefore \quad x = x' \cos\theta - y' \sin\theta,$$

$$y = x' \sin\theta + y' \cos\theta ;$$

$$\therefore \quad (\Sigma mxz) = \Sigma (mx'z) \,.\, \cos\theta - \Sigma (my'z) \,.\, \sin\theta = 0,$$

$$\Sigma (myz) = \Sigma (mx'z) \,.\, \sin\theta + \Sigma (my'z) \,.\, \cos\theta = 0.$$

It does not follow, however, that $\Sigma (mxy) = 0$ for every system of axes of x and y. The truth of this additional equation then distinguishes the system of principal axes from any other.

16. The determination of the principal axes of a given body at a proposed point is identical with finding the position of the axes of figure of an ellipsoid whose general equation referred to its centre is presented. Since, however, when these axes are employed with practical advantage, one of them can in most of such cases be assigned by inspection from the form of the body, the following proposition is then useful to complete the determination of the system.

17. Given one principal axis of a body at a proposed point, to find the other two.

Let the body be referred to a system of rectangular axes at the proposed point so that the given principal axis is the axis of z. The required principal axes lie in the plane xy. Let them make an angle θ with the axes of x and y respectively, and let x', y' be the co-ordinates of a particle m referred to them.

Then $$x' = x \cos\theta + y \sin\theta,$$

$$y' = - x \sin\theta + y \cos\theta ;$$

$$\therefore \quad 0 = \Sigma (mx'y') = \Sigma (mxy) \,.\, \cos 2\theta + \tfrac{1}{2} \Sigma m (y^2 - x^2) \,.\, \sin 2\theta.$$

$$\therefore \quad \tan 2\theta = \frac{2\Sigma (mxy)}{\Sigma m (x^2 - y^2)} .$$

Also $\Sigma(mx'z') = 0$, $\Sigma(my'z') = 0$; otherwise no system of principal axes would exist whereof the axis of z is one. (15).

The values of θ which the preceding equation assigns give the same pair of principal axes presented in a different order, unless the expression is indeterminate, when $\Sigma(mxy) = 0$, $\Sigma(mx^2) = \Sigma(my^2)$, and then $0 = \Sigma(mx'y')$ is satisfied independently of θ, so that any pair of axes in the plane xy forms a system of principal axes with z.

Obs. The angle θ is measured from the positive direction of the axis of x towards the corresponding portion of the axis of y.

18. Hence whenever the form of the body allows an axis of z to be taken for which $\Sigma(mxz) = 0$, $\Sigma(myz) = 0$, whatever be the axes of x and y, then the other two principal axes at the origin can at once be assigned.

19. A straight line perpendicular to the plane of a lamina will be a principal axis at the point where it meets the lamina.

20. The axis of a solid of revolution bounded by planes perpendicular to this axis is a principal axis at any point in its length.

21. Def. The moments of inertia of a body about its principal axes at any point are called its principal moments at that point.

If the principal axes be co-ordinate axes, A, B, C the principal moments, and Q the moment of inertia about any axis through the origin whose direction cosines are α, β, γ,

$$Q = A\alpha^2 + B\beta^2 + C\gamma^2.$$

Hence when the moments of inertia of a body about its principal axes at any point are known, its moment of inertia about any axis through that point results.

22. If an ellipsoid be constructed about the origin as centre whose principal semi-axes are $A^{-\frac{1}{2}}$, $B^{-\frac{1}{2}}$, and $C^{-\frac{1}{2}}$, a radius vector of this ellipsoid is the inverse square root of the

moment of inertia of the body about the direction of that radius vector. This is called the central ellipsoid of the body at the origin adopted.

23. Let the principal moments be unequal.

Since $Q = A\alpha^2 + B\beta^2 + C\gamma^2$,

and $1 = \alpha^2 + \beta^2 + \gamma^2$;

$$\therefore \quad 0 = (A - Q)\alpha^2 + (B - Q)\beta^2 + (C - Q)\gamma^2.$$

Hence of the quantities $A - Q$, $B - Q$, $C - Q$, one must always have a contrary sign to the other two, or Q is between the greatest and least of the magnitudes A, B, C.

Hence of the moments of inertia of a body about axes through a given point, the moment of inertia about one of the principal axes is greatest and about another is least.

Conversely, axes about which the moment of inertia is greatest or least are principal axes.

The axes, about which the moment of inertia has a given magnitude, lie on a cone of the second order with the origin as vertex. (Gregory's Solid Geometry, 80.)

24. If two of the principal moments are equal, if $A = B$, for instance,

$$Q = A(1 - \gamma^2) + C\gamma^2;$$

therefore Q depends on γ only, or the moment of inertia is the same about all axes which lie on a right circular cone about that principal axis with respect to which the moment of inertia is not equal to those about the other two.

25. If all the principal moments of inertia are equal, the moment of inertia of the body about all axes through the origin is the same.

SECTION II.

D'ALEMBERT'S PRINCIPLE.

---

26. In examining the motion of a body of finite size we have first to consider the geometrical character of its motion, and thus to fix upon certain elements of that motion, from the determination of which the position, direction, and velocity of any proposed point of the body will at any instant be known. These elements of the motion will be selected according to the particular circumstances of the body, and the problem of determining the body's motion will consist in so connecting these with the given forces which are in action upon it, that by a knowledge of the latter and the state of the body at some epoch, the former can at once be assigned.

27. Now the causes which influence the motion of each particle of a rigid body are, 1^0. forces extraneous to the body, and 2^0. forces resulting from its connection with the other particles of the body, and called in consequence molecular. Of these latter we know neither the law nor the intensity, and therefore the computation of the motion of the body by calculating the motions of its individual particles is impossible. This difficulty is eluded by the principle which follows, called D'Alembert's principle.

28. The dynamical circumstances of the motion of a particle shall first be reviewed.

I. Let the forces acting on the particle be finite, or such as can produce no change in its state except in an appreciable time.

1. If x, y, z be the rectangular co-ordinates of the

particle's position at time t from a fixed epoch, the accelerating forces on it in directions of these co-ordinates are

$$\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2}, \quad \frac{d^2z}{dt^2};$$

and if m be the mass of the particle, the moving forces upon it in the same directions are

$$m\frac{d^2x}{dt^2}, \quad m\frac{d^2y}{dt^2}, \quad m\frac{d^2z}{dt^2}.$$

The directions of the forces are those in which the co-ordinates are measured, independently of the direction of the particle's motion.

2. If v be the velocity of the particle at time t when it has described along its path a length s from a fixed point thereof, and ρ the radius of absolute curvature of its path at its present position, the accelerating forces upon it are

$\frac{d^2s}{dt^2}$ in direction of its motion,

$\frac{v^2}{\rho}$ in the normal and towards the centre of curvature.

These multiplied by m give the moving forces in the same directions.

3. If r, θ be the polar co-ordinates of a particle moving in one plane, the accelerating forces upon it resolved in and perpendicular to the radius vector are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \text{ and } r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\cdot\frac{d\theta}{dt},$$

the former estimated *from* the pole, the latter in the direction in which θ is measured. These, as before, may be converted into the moving forces in the same directions.

Adopting then any one of these three modes of resolution, we have analytical expressions for the accelerating and the

moving forces which must act on a particle in motion, supposing them finite.

II. If the forces acting on the particle be impulsive, or such as can produce a finite alteration in its state in an inappreciable time, then if the particle has velocities u, v, w, in directions of the three rectangular co-ordinate axes, and if these suddenly become u', v', w', the force upon it consists of impulsive forces in directions of the same axes whose numerical representations are

$$u' - u,\ v' - v,\ w' - w,$$

$$\text{or } m(u' - u),\ m(v' - v),\ m(w' - w),$$

according as they are estimated with respect to the velocity only, or to the momentum which they can produce, i.e. as accelerating or moving impulsive forces.

29. DEF. Extraneous forces acting on a rigid body are called impressed forces.

DEF. The force under which a particle of a body if free might move as it really moves is called the effective force on that particle.

Of this force the analytical representations have just been given (28).

30. D'Alembert's Principle.

If the effective moving forces belonging to the several particles of a body be applied to them respectively in directions contrary to the directions of the same forces, the conditions of statical equilibrium of the body are satisfied if it be under the action of these supposed forces and the impressed forces which are really applied to it.

31. Ex. A wheel is revolving uniformly about its centre. Let v be the velocity of a particle of it at distance r from the centre. Then since the particle is describing a circle with uniform velocity v, the force which if it were free must act upon it that

it might have this motion is $\frac{v^2}{r}$ towards the centre of the wheel. This then is the effective accelerating force on the particle in question regarded as a molecule of the rigid body. In this instance the effective force on each particle is wholly perpendicular to the direction of its motion.

Ex. 2. If a body is moving in such a manner that the direction of each particle remains unchanged, the effective force on each particle is in direction of its motion. If the motion of each particle remains constant as to velocity as well as direction, the effective force on each particle is zero.

Ex. 3. If a wheel be sliding uniformly along its axis and also revolving uniformly about it, the effective force on a molecule is $\frac{v^2}{r}$ acting perpendicularly to the axis, r being the particle's distance from the axis, and v the part of its velocity perpendicular to the axis.

Ex. 4. If a body have its motion so altered by impulses that the velocity v of a particle m is suddenly changed to v', its direction being unaltered, the effective impulsive force on the particle is $v' - v$ or $m(v' - v)$ the force being estimated as an accelerating or moving force respectively.

32. Thus the principle of D'Alembert is the means of evading the unknown molecular forces which unite the parts of a rigid body. It may thus be illustrated. If P (fig. 3) be a molecule of a rigid body, I the impressed force upon it, and M the molecular force arising from the connection of P with contiguous parts of the body, then E the resultant of these will be the effective force on P. If E' be equal and opposite to E, the assertion of D'Alembert's principle is that the system of forces I and the system of forces E' taken throughout the rigid body satisfy the conditions of statical equilibrium. The forces I and E' will not generally balance one another on any assigned particle, but when the whole body is contemplated, the aggregate of the forces M may be removed.

33. The proof of the principle of D'Alembert will rest, like the laws of motion of a molecule, on the verification by observation of nature, of results obtained from calculations based upon it. The continued accordance of complex results as observed and as predicted by calculation is our security that the principle on which our calculations have proceeded cannot be untrue.

34. When the body degenerates into a particle, it will be seen that the three laws of a particle's motion are included in this principle. It is an extension of those laws, and like them depends for its certainty on the accordance of calculations in which it is assumed with observations of nature.

35. D'Alembert's principle has been stated without respect to the manner in the which particles of the body under consideration are connected or act on one another. Therefore although at present the principle is only employed in examining the motion of a rigid body, it must be recollected to be no less true of an extensible, elastic or fluid substance.

SECTION III.

THE MOTION OF A RIGID BODY ABOUT A FIXED AXIS.

36. GEOMETRICAL nature of the motion of a rigid body about a fixed axis.

When a body revolves about a fixed axis, every point of it describes a circle about that axis in a plane perpendicular to the axis. The position of the body at any instant is designated by the angle which a plane fixed in the body and parallel to the axis of rotation makes with a plane fixed in space and also parallel to the same axis.

37. The angular velocity of the body is the rate at which this angle changes. Its amount at any instant is measured, 1°. when it is uniform, by the angle described in any unit of time, 2°. when it is variable, by the angle which would be described in an unit of time if the body retained through that unit the angular motion which it has at the instant considered.

38. Let ω be the angular velocity.

(1) If the angular velocity be constant we mean that the body revolves through an angle ω in the unit of time, and the angle is described equably; therefore ω is the ratio of the angle described in any time to that time.

(2) If the angular velocity be variable, let θ be the angle described by the body at time t, $\delta\theta$ the angle described thence in time δt at the end of which the angular velocity becomes $\omega + \delta\omega$. Then if δt be sufficiently small

$$\delta\theta \text{ lies between } \begin{matrix} \omega \,.\, \delta t, \\ (\omega + \delta\omega)\,\delta t; \end{matrix}$$

$$\therefore \frac{\delta\theta}{\delta t} \text{ lies between } \frac{\omega}{\omega + \delta\omega};$$

$\therefore$ in the limit when $\delta\omega$ vanishes

$$\omega = \text{limit } \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt}.$$

Obs. The direction in which θ is measured is fixed in space and independent of the direction in which the body is revolving. The body's motion will be in or contrary to the direction in which θ is measured according as $\frac{d\theta}{dt}$ is positive or negative.

39. If ω be the angular velocity of the body, the velocity of a point at distance r from the axis is ωr in a direction perpendicular both to that distance and the axis. Hence if ω is determined, the velocity of every particle is known. Also if the position of the assumed plane in the body be assigned, then by the geometrical form of the body the position in space of every point of it is also assigned. The position therefore of this plane and the angular velocity are the elements by which the body's dynamical state is determined.

40. If the body has an angular velocity ω about the axis of z in a direction from the positive part of the axis of x towards that of y, the velocity of a particle whose co-ordinates are x, y is $-\omega y$ parallel to x, and ωx parallel to y.

Motion of a body about an axis under finite forces.

41. To determine the motion of a rigid body revolving about a fixed axis under given forces.

Let the axis about which the body revolves be made the axis of z, and let x, y, z be the co-ordinates at time t of a particle of the body of mass m, r its distance from the axis of z, and θ the inclination of r to the plane xz, X, Y, Z the impressed accelerating forces on this particle parallel to the

co-ordinate axes. Then the effective moving forces on this particle are $mr\left(\frac{d\theta}{dt}\right)^2$ in direction of r, and $mr\frac{d^2\theta}{dt^2}$ perpendicular to r and in the direction in which θ is measured (28). If such effective forces belonging to each particle be supposed to act on it in directions contrary to their actual directions, the conditions of statical equilibrium are satisfied between these and the impressed forces on the body, among which latter the reaction of the axis is included.

Hence if moments be taken about the axis of z,

$$0 = \Sigma\{m(Yx - Xy)\} - \Sigma\left(mr^2 . \frac{d^2\theta}{dt^2}\right),$$

the summation extending through all particles of the body.

Now let a fixed plane in the body parallel to z be inclined to the plane of xz at an angle ϕ measured in the same direction with θ, and let $\theta = \phi + \alpha$, where α depends on the position in the body of the molecule considered and so by the condition of rigidity is independent of t; $\therefore \frac{d^2\theta}{dt^2} = \frac{d^2\phi}{dt^2}$. And since ϕ is independent of any particular particle,

$$\frac{d^2\phi}{dt^2} = \frac{\Sigma m(Yx - Xy)}{\Sigma(mr^2)}.$$

Hence if the forces acting on each particle of the body in any position, excluding the reaction of the axis, be given, and so be known functions of ϕ and the position of the molecules in the body, since the reaction of the axis has no moment about the axis and does not appear in the latter member of this equation, integration will give $\frac{d\phi}{dt}$, and then ϕ in terms of t, and so determine the velocity and position of the body at any proposed instant.

42. To find the pressure on the axis of revolution.

Since any system of forces acting on a rigid body is at most reducible to two single forces, let the pressure on the

axis be reduced to a force at the origin and another at a known distance a in the positive direction of the axis of z. Let F, G, H be components of the former parallel to the co-ordinate axes, F', G', H' similar components of the latter. Forces equal and opposite to these are the reactions of the axis upon the body. The equations of equilibrium give

$$\Sigma(mX) - F - F' = -\Sigma\left\{mr\cos\theta\left(\frac{d\theta}{dt}\right)^2\right\} - \Sigma\left(mr\sin\theta\cdot\frac{d^2\theta}{dt^2}\right),$$

$$\Sigma(mY) - G - G' = \Sigma\left(mr\cos\theta\frac{d^2\theta}{dt^2}\right) - \Sigma\left\{mr\sin\theta\left(\frac{d\theta}{dt}\right)^2\right\},$$

$$\Sigma(mZ) - H - H' = 0.$$

$$\Sigma m(Zy - Yz) + G'a = -\Sigma\left\{mzr\cos\theta\frac{d^2\theta}{dt^2}\right\} + \Sigma\left\{mzr\sin\theta\cdot\left(\frac{d\theta}{dt}\right)^2\right\},$$

$$\Sigma m(Xz - Zx) - F'a = -\Sigma\left\{mzr\cos\theta\left(\frac{d\theta}{dt}\right)^2\right\} - \Sigma\left\{mzr\sin\theta\frac{d^2\theta}{dt^2}\right\}.$$

The sixth equation of equilibrium has been already employed in (41).

If $\bar{x}$, $\bar{y}$ be the co-ordinates of the centre of gravity of the body, M its mass, these equations become

$$\Sigma(mX) - F - F' = -M\bar{x}\left(\frac{d\phi}{dt}\right)^2 - M\bar{y}\frac{d^2\phi}{dt^2},$$

$$\Sigma(mY) - G - G' = M\bar{x}\frac{d^2\phi}{dt^2} - M\bar{y}\left(\frac{d\phi}{dt}\right)^2,$$

$$\Sigma(mZ) - H - H' = 0.$$

$$\Sigma m(Zy - Yz) + G'a = -\Sigma(mxz)\frac{d^2\phi}{dt^2} + \Sigma(myz)\left(\frac{d\phi}{dt}\right)^2,$$

$$\Sigma m(Xz - Zx) - F'a = -\Sigma(mxz)\left(\frac{d\phi}{dt}\right)^2 - \Sigma(myz)\cdot\frac{d^2\phi}{dt^2}.$$

From the third equation the sum of H and H' is known, their separate values under this condition being indeterminate.

The other four equations give F, G, F', G', since $\frac{d^2\phi}{dt^2}$, and thence $\left(\frac{d\phi}{dt}\right)^2$ are already known and the expressions $\Sigma(mxz)$, $\Sigma(myz)$ can be evaluated by integration according to the known form of the body.

43. If no force be impressed on the body besides the reaction of the axis, $\frac{d^2\phi}{dt^2} = 0$, $\frac{d\phi}{dt} = \omega$ is constant, and

$$F + F' = -M\bar{x}\omega^2,$$
$$G + G' = M\bar{y}\omega^2,$$
$$H + H' = 0,$$
$$G'a = \Sigma(myz)\omega^2,$$
$$F'a = \Sigma(mxz)\omega^2.$$

Two particular cases deserve notice:

(1) If the fixed axis pass through the centre of gravity of the body the former three equations shew that the pressure on the axis forms a couple.

(2) If the fixed axis be a principal axis, since the assumed axes of x and y may be so taken as to coincide with other principal axes of the body at the instant considered, so that $\Sigma(mxz) = 0$, $\Sigma(myz) = 0$, the latter two equations shew that the pressure on the axis is reducible to a single force.

If the conditions subsist together, there is no pressure on the axis.

44. Forces parallel to the axis of rotation do not influence the motion of the body, but affect the pressure on the axis.

45. If the symmetry of the body or any other cause indicates that the pressure on the axis is reducible to a single

force, the preceding investigation has a corresponding simplification.

46. If the whole pressure on the axis be reduced to the two R and S, the former R in a plane through the centre of gravity and from the axis, and the latter S perpendicular to this plane, and in the direction in which ϕ the inclination of this plane to the horizon is measured, and if P, Q be the impressed forces in the same directions, $\bar{r}$ the distance of the centre of gravity from the axis

$$R = P + M\bar{r}\left(\frac{d\phi}{dt}\right)^2,$$

$$S = Q - M\bar{r}\frac{d^2\phi}{dt^2}.$$

47. The indeterminateness which arises in the pressure on the axis in direction of its length even when the points are assigned where we suppose it applied, like a similar indeterminateness in the equilibrium of a rigid body capable of revolving about an axis, is contrary to experience; for we know that if a solid body be suspended at two points so as to be capable of turning about the straight line which joins them as an axis, the pressure on each of these points when the body is at rest or in motion is under similar circumstances fixed and determinate. The contradiction which practice gives to the theoretical result arises from the imperfect rigidity of the bodies which are under our observation. The law according to which pressure is distributed through bodies of this kind which do not completely fulfil the conception of a rigid body on which our calculations have proceeded, would supply if it were within the reach of mathematical calculation, another equation of condition, and make the two pressures determinate of which in the case of a perfectly rigid body the sum only is assigned. (Poisson. III. 1.)

48. When a body rotates about a horizontal axis under the action of gravity alone, the case deserves a separate inves-

tigation on account of the importance of its practical applications.

Let h be the distance of the centre of gravity from the axis, ϕ the angle at which this distance is inclined to the horizon at time t. The effective moving forces on a particle of mass m being $mr\left(\frac{d\phi}{dt}\right)^2$ and $mr\frac{d^2\phi}{dt^2}$ in and perpendicular to its distance r from the axis, the equation of moments about the axis of rotation gives

$$0 = Mgh\cos\phi - \Sigma(mr^2)\frac{d^2\phi}{dt^2}.$$

If Mk^2 be the moment of inertia of the body about an axis through its centre of gravity parallel to the axis of rotation,

$$\Sigma(mr^2) = M(k^2 + h^2),$$

$$\text{and } \frac{d^2\phi}{dt^2} = \frac{gh\cos\phi}{k^2 + h^2}.$$

This is the equation of motion of a particle hung by an inextensible thread of length $\frac{k^2 + h^2}{h}$, and swinging under the action of gravity. For this reason, if in the line h produced a point be taken at a distance $\frac{k^2 + h^2}{h}$ from the axis, this point is called the centre of oscillation of the body with respect to this axis. The following is its definition:

DEF. When a rigid body moves about a fixed horizontal axis under the action of gravity, in the straight line which is drawn through the centre of gravity perpendicular to the axis a point can be found such that if the mass of the body were collected there and hung by a thread from the axis, the angular motion of the point would, under the same initial circumstances, be the same with that of the body, and this point is called the centre of oscillation of the body with respect to the axis.

49. Hence when a body makes small oscillations about a fixed horizontal axis, it is only necessary to calculate the value of the expression $\frac{k^2+h^2}{h} = l$ suppose, and the time of a small oscillation is $\pi\sqrt{\frac{l}{g}}$. The double of this is the interval in which the body returns to the same place in the same state of motion.

In adopting the approximate expression $\pi\sqrt{\frac{l}{g}}$ as the time of oscillation the proportionate error is of the order $\left(\sin\frac{\alpha}{2}\right)^2$, where 2α is the whole angle through which the pendulum vibrates. (Poisson. II. 5). By this test the extent of vibration for which the approximate time of vibration may be taken can in any proposed case be assigned. If the vibration, for instance, on each side of the vertical is within 5^0, the error will not exceed $\frac{1}{2000}^{\text{th}}$ part of the time of oscillation.

50. If h' be the distance of the centre of oscillation from the centre of gravity when h is the distance of the axis,

$$h' = \frac{k^2+h^2}{h} - h = \frac{k^2}{h}, \text{ or } hh' = k^2.$$

Hence the body will oscillate about a parallel axis through the centre of oscillation in the same time as it oscillates about the original axis, the extent of vibration being the same.

51. If in any straight line through the centre of gravity perpendicular to the direction to which k has reference points be taken at distances h, h' from the centre of gravity, and on opposite sides of it, such that $hh' = k^2$, then the length of the simple equivalent pendulum when an axis in the said direction passes through either of these points is $h+h'$; so that the time of vibration about each of these two axes is the same, and the length of the equivalent simple pendulum is in each case the distance between the axes. Each of these points is the centre

of oscillation in reference to the other as a centre of suspension. The centres of oscillation and suspension are therefore convertible; and if two parallel axes of a body can be found about which the time of a small oscillation is the same, the distance between them, if it pass through the centre of gravity of the body, is the length of the simple isochronous pendulum.

52. The convertibility of the centres of oscillation and suspension was employed by Captain Kater in finding the length of a simple pendulum vibrating seconds, and consequently the force of gravity at the place of observation. It is impossible to find the centre of oscillation of a body by calculation with sufficient accuracy for this purpose by reason of inequalities of form and structure which no workmanship can obviate. The position of that centre was therefore obtained by observation in the following manner.

A bar of brass one inch and a half wide and one eighth of an inch thick was pierced by two holes through which triangular wedges of steel called knife edges were inserted, so that the pendulum could vibrate on the edge of either of these as an axis. On either of these the pendulum could be hung between two fixed horizontal agate plates upon which it then vibrated. The axes were about 39 inches apart. Three weights were attached to the bar. One, much the largest, was immoveable, fixed near one knife edge and beyond it. The other two were between the knife edges, and capable of small motions along the bar by screws. These latter were so adjusted by trial that the time of a small vibration through an angle of about 1^0 was the same when either knife edge was the axis, so that each gave the centre of oscillation belonging to the other. The distance of the knife edges was obtained by placing the pendulum so that the edges were viewed by two fixed microscopes, each furnished with a micrometer of ascertained value, and afterwards placing a scale of known accuracy in a similar position under the same microscopes. If l be this length, t the time of vibration about either axis, then

$$t = \pi \sqrt{\frac{l}{g}}, \text{ or } g = l . \frac{\pi^2}{t^2},$$

and the length of the seconds pendulum is $\frac{g}{\pi^2}$ in the unit of length in which g is expressed.

The time of oscillation of the pendulum was thus found. The pendulum was suspended in front of a clock pendulum, oscillating in very nearly the same time, so that both when at the lowest points of their oscillation were in the field of view of a fixed telescope. The times by the clock were noted when the two pendulums coincided so that one covered the other in their passage over the field. The rates of the two were so nearly equal that between two such successive times one pendulum had gained a complete oscillation upon the other; and then the number of oscillations of the clock pendulum in this interval being known by the clock face, the number of the oscillations of the other was also known, and consequently its time of vibration. Phil. Trans. 1818.

53. A determination by experiment of the length of a simple pendulum vibrating seconds has two important uses.

(1) It gives a value from experiment of the force of gravity at a given place, and thus supplies a test of theories respecting the constitution of the earth whereby a value of the force of gravity is computed.

(2) It has been adopted as a standard of length on account of being invariable and capable at any time of recovery. An act of parliament, 5 Geo. IV. defines the yard to contain 36 such parts, of which parts there are 39·1393 in the length of a pendulum vibrating seconds of mean time in the latitude of London in vacuo at the level of the sea at temperature 62 F. The commissioners, however, appointed to consider the mode of restoring the standards of weight and measure which were lost by fire in 1834, report that several elements of reduction of pendulum experiments are yet doubtful or erroneous, so that the results of a convertible pendulum are not so trustworthy as to serve for supplying a standard of length; and they recommend a material standard, the distance

namely between two marks on a certain bar of metal under given circumstances, in preference to any standard derived from measuring phenomena in nature. (Report, 1841.)

54. Since metals expand as their temperature increases, the position of the centre of oscillation and consequently the time of vibration through a given arc of a metallic pendulum will change with the temperature of the atmosphere. In order to employ metallic pendulums in clocks where uniformity of the time of their vibration is essential, contrivances are employed whereby the expansion of different parts so counteract the effect of one another as to leave the centre of oscillation at a nearly constant distance from the axis of suspension and the time of the pendulum's vibration in consequence permanent.

55. The compensation pendulum of Harrison called from its form the gridiron pendulum (fig. 4) consists of five vertical rods of steel and four of brass placed alternately, and united by horizontal rods of brass. The central rod supports a disc or lens of metal called the bob of the pendulum. The pendulum, as it is figured, vibrates in the plane of the paper.

The principle on which the compensation is produced may thus be explained. In a system of parallel metallic rods in figure 5 let A be the fixed point about which the pendulum vibrates, G the bob. If CD alone expand in consequence of an increase of temperature, the points D, E and consequently G are raised. If AB and EF were alone to expand, G would sink by the increase of length of each of these rods. Hence if DC be of brass, and AB, EF of steel whose expansibility is little more than half that of brass, it is possible that the upward expansion of the brass rod may so correct the downward expansions of the steel rods as to leave the centre of oscillation nearly unaffected. Rods of brass and steel united on this principle form the gridiron pendulum of Harrison.

56. A rod of brass is increased by ·00018782 of its length, and one of steel by ·00010791 of its length when its tempera-

ture is increased by 1^0 centigrade. (Annuaire par le bureau des longitudes, 1846). These at least are mean values, from which, however, the expansibilities of different specimens of these metals may differ so far as to demand an independent determination in individual cases whenever accuracy is required.

57. The mercurial pendulum of Graham is a vertical cylinder of glass or cast iron containing mercury, and suspended by a steel rod. By the expansion of the steel rod on an increase of temperature the centre of oscillation is lowered while by the expansion of the mercury in the cylinder it is elevated. Since mercury expands much faster than steel, the quantity of mercury may be so adjusted that the centre of oscillation is nearly permanent under ordinary changes of temperature.

The mercurial pendulum has this advantage over that of Harrison, that it admits of more easy adjustment by a screw affecting the attachment of the cylinder, by which the distance of the cylinder from the point of suspension may be altered.

Motion of a body about an axis under impulsive forces.

58. To determine the change in the motion of a rigid body revolving about a fixed axis after given impulses have been applied to it.

Let the axis about which the body is revolving with an angular velocity ω be made the axis of z; let x, y, z be the rectangular co-ordinates of a particle of the body whose mass is m, r its distance from the axis, mX, mY, mZ the impressed impulses upon it parallel to the co-ordinate axes. If ω' be the initial angular velocity of the body after the impulse, the effective impulsive force on that particle is $mr(\omega' - \omega)$ in direction of its motion; and if such effective forces on the several particles of the body be supposed to act on them in directions contrary to their own, between such forces and the impressed forces the conditions of equilibrium of the body are satisfied.

$$\therefore\ 0 = \Sigma m(Yx - Xy) - \Sigma mr^2(\omega' - \omega);$$

$$\text{or}\quad \omega' - \omega = \frac{\Sigma m(Yx - Xy)}{\Sigma(mr^2)}.$$

Since the impulsive reactions of the axis have no moment about it and so do not appear on the latter side of the equation, ω' is known if the impressed impulsive forces be given.

59. To find the impulse on the axis.

Since the impulses on the axis can be reduced to two forces at most, let these act at the origin and at an assumed distance a in the positive direction of the axis of z, and let F, G, H be components of the former, F', G', H' of the latter. Then the equations of equilibrium give

$$0 = \Sigma(mX) - F - F' + \Sigma my(\omega' - \omega),$$

$$0 = \Sigma(mY) - G - G' - \Sigma mx(\omega' - \omega),$$

$$0 = \Sigma(mZ) - H - H',$$

$$0 = \Sigma m(Zy - Yz) + G'a + \Sigma mxz(\omega' - \omega),$$

$$0 = \Sigma m(Xz - Zx) - F'a + \Sigma myz(\omega' - \omega),$$

the sixth equation being that employed in (58).

If $\bar{x}$, $\bar{y}$ be co-ordinates of the centre of gravity of the body, M its mass, these equations become

$$0 = \Sigma(mX) - F - F' + M\bar{y}(\omega' - \omega),$$

$$0 = \Sigma(mY) - G - G' - M\bar{x}(\omega' - \omega),$$

$$0 = \Sigma(mZ) - H - H',$$

$$0 = \Sigma m(Zy - Yz) + G'a + (\omega' - \omega)\Sigma(mxz),$$

$$0 = \Sigma m(Xz - Zx) - F'a + (\omega' - \omega)\Sigma(myz).$$

From the third equation the sum of H and H' is found, their separate values under this condition remaining unknown.

The other four equations give F, F', G, G'. The explanation given by Poisson (47) applies to this anomaly.

60. If a body at rest, but capable of turning about a fixed axis, can be so struck that there may be no impulse on the axis, a point where the impulse is applied to the body in this case is called a centre of percussion.

Let the axis about which the body can revolve be the axis of z, and let x, y, z be the rectangular co-ordinates of a particle whose mass is m, X, Y, Z the components of the impulse acting on the body at the point $x'y'z'$. Then this impulse must be at equilibrium with the effective impulsive force on each particle applied in a reversed direction, which if ω be the body's initial velocity will be $-m\omega r$, where r is the particle's distance from the axis of z.

Hence
$$0 = X + \Sigma(m\omega y),$$
$$0 = Y - \Sigma(m\omega x),$$
$$0 = Z,$$
$$0 = Zy' - Yz' + \Sigma(m\omega xz),$$
$$0 = Xz' - Zx' + \Sigma(m\omega yz),$$
$$0 = Yx' - Xy' - \Sigma(m\omega r^2).$$

From the third equation $Z = 0$, or the impulse on the body must be perpendicular to the axis about which it can rotate. There then remain five equations to determine x', y', z' and ω, which will only in certain cases be consistent, or in certain cases only and not in all will a centre of percussion exist.

From these equations

$$z' = -\frac{\omega\Sigma(myz)}{X} = \frac{\Sigma(myz)}{\Sigma(my)},$$

$$z' = -\frac{\omega \Sigma (mxz)}{Y} = \frac{\Sigma (mxz)}{\Sigma (mx)}.$$

If the body be so constituted that these are equal to one another, either of them gives z' and then the last equation

$$0 = x' \Sigma (mx) + y' \Sigma (my) - \Sigma (mr^2)$$

gives a straight line parallel to the plane of xy, any point of which rigidly connected with the body is a centre of percussion.

SECTION IV.

MOTION OF A RIGID BODY ABOUT A FIXED POINT.

61. GEOMETRICAL nature of the motion of a body about a fixed point.

A conception of the motion of a rigid body about a fixed axis having been already acquired, the motion of a body about a fixed point will be reduced to this and explained by it. When a rigid body moves so that one point of it is fixed in space, every particle of the body retains an invariable distance from that point. This is the geometrical condition of the motion. It will be shewn that at every instant a certain straight line can be assigned, by rotation about which as if it were fixed the motion of the body at that instant is properly represented. This line is called an instantaneous axis.

62. In problems of this class there will generally be occasion to introduce two systems of rectangular co-ordinate axes, both originating in the fixed point, one system fixed in space, another fixed in the body and moveable with it.

Let a particle m have co-ordinates x, y, z referred to the former system, x', y', z' referred to the latter. Then x', y', z' are geometrical quantities independent of the time, and only altered in passing from one point of the rigid body to another; but x, y, z are dynamical elements and functions of the time, designating the position of m in space, and so varying with different positions of the body, although the same particle of it be kept under contemplation.

If the position of the axes of x', y', z', or of any two assumed lines in the body, be assigned with respect to the axes of x, y, z, the position of the body is determined.

63. To prove that the motion of a body about a fixed point may be exhibited at any instant by rotation about an axis.

(1) The fixed point being the origin, let x, y, z be the rectangular co-ordinates of a particle of the body referred to axes fixed in space, x', y', z' co-ordinates of the same particle referred to rectangular axes fixed in the body and moveable with it;

$$\therefore \cdot \quad x' = a_1 x + b_1 y + c_1 z,$$

$$y' = a_2 x + b_2 y + c_2 z,$$

$$z' = a_3 x + b_3 y + c_3 z;$$

where the coefficients are functions of the angles which the moveable co-ordinate axes make with the fixed axes and are thus functions of the time, but they respect only the position of the body in space, and not any particular particle of it, and are connected by the conditions

$$a_1^2 + a_2^2 + a_3^2 = 1 = b_1^2 + b_2^2 + b_3^2 = c_1^2 + c_2^2 + c_3^2,$$

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = 0, \quad a_1 c_1 + a_2 c_2 + a_3 c_3 = 0, \quad b_1 c_1 + b_2 c_2 + b_3 c_3 = 0. \quad \text{(A)}.$$

Now, $$0 = a_1 \frac{dx}{dt} + b_1 \frac{dy}{dt} + c_1 \frac{dz}{dt} + x \frac{da_1}{dt} + y \frac{db_1}{dt} + z \frac{dc_1}{dt}, \qquad (1)$$

$$0 = a_2 \frac{dx}{dt} + b_2 \frac{dy}{dt} + c_2 \frac{dz}{dt} + x \frac{da_2}{dt} + y \frac{db_2}{dt} + z \frac{dc_2}{dt}, \qquad (2)$$

$$0 = a_3 \frac{dx}{dt} + b_3 \frac{dy}{dt} + c_3 \frac{dz}{dt} + x \frac{da_3}{dt} + y \frac{db_3}{dt} + z \frac{dc_3}{dt}. \qquad (3)$$

$a_1 . (1) + a_2 . (2) + a_3 . (3)$ gives by virtue of (A)

$$0 = \frac{dx}{dt} + y\left(a_1 \frac{db_1}{dt} + a_2 \frac{db_2}{dt} + a_3 \frac{db_3}{dt}\right) + z\left(a_1 \frac{dc_1}{dt} + a_2 \frac{dc_2}{dt} + a_3 \frac{dc_3}{dt}\right).$$

Let $$b_1\frac{dc_1}{dt}+b_2\frac{dc_2}{dt}+b_3\frac{dc_3}{dt}=\omega_x=-\left(c_1\frac{db_1}{dt}+c_2\frac{db_2}{dt}+c_3\frac{db_3}{dt}\right),$$

$$c_1\frac{da_1}{dt}+c_2\frac{da_2}{dt}+c_3\frac{da_3}{dt}=\omega_y=-\left(a_1\frac{dc_1}{dt}+a_2\frac{dc_2}{dt}+a_3\frac{dc_3}{dt}\right),$$

$$a_1\frac{db_1}{dt}+a_2\frac{db_2}{dt}+a_3\frac{db_3}{dt}=\omega_z=-\left(b_1\frac{da_1}{dt}+b_2\frac{da_2}{dt}+b_3\frac{da_3}{dt}\right).$$

$$\therefore\ \frac{dx}{dt}=\omega_y z-\omega_z y,$$

So $$\frac{dy}{dt}=\omega_z x-\omega_x z,$$

$$\frac{dz}{dt}=\omega_x y-\omega_y x$$

Hence to determine points which if they be supposed rigidly united with the body are at rest at the instant under consideration, we have

$$0=\omega_y z-\omega_z y,$$

$$0=\omega_z x-\omega_x z,$$

$$0=\omega_x y-\omega_y x.$$

These equations not being independent but reducible to the two, $\frac{x}{\omega_x}=\frac{y}{\omega_y}=\frac{z}{\omega_z}$, give a straight line through the origin as the locus of points which have no velocity.

(2) The direction of the motion of a point x, y, z is given by the equations

$$\frac{X-x}{\omega_y z-\omega_z y}=\frac{Y-y}{\omega_z x-\omega_x z}=\frac{Z-z}{\omega_x y-\omega_y x}.$$

Since $\omega_x(\omega_y z-\omega_z y)+\omega_y(\omega_z x-\omega_x z)+\omega_z(\omega_x y-\omega_y x)=0$, this direction is perpendicular to the straight line which has been already determined as the locus of particles at rest.

This only proves that the direction of the
[illegible]cles motion is at any instant in
[illegible]lane to wh. the instantaneous
[illegible] is at rest

(3) If ρ be the distance of a particle x, y, z from the line $\frac{X}{\omega_x} = \frac{Y}{\omega_y} = \frac{Z}{\omega_z}$,

$$\rho^2 = x^2 + y^2 + z^2 - \frac{(x\omega_x + y\omega_y + z\omega_z)^2}{\omega_x^2 + \omega_y^2 + \omega_z^2},$$

and the velocity of the same particle

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= \sqrt{(\omega_x^2 + \omega_y^2 + \omega_z^2)(x^2 + y^2 + z^2) - (x\omega_x + y\omega_y + z\omega_z)^2}$$

$$= \rho\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} = \rho\omega.$$

From these three results it appears that at the instant considered the motion of the body is the same as if it were rotating about an axis through the fixed point with an angular velocity ω. (39).

Since the quantities ω_x, ω_y, ω_z are in general variable, the position of the instantaneous axis and the angular velocity about it change in general in successive instants.

64. Equations (A) can be transformed into the following system:

$$a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2 = a_3^2 + b_3^2 + c_3^2 = 1,$$

$$a_2a_3 + b_2b_3 + c_2c_3 = 0,$$

$$a_1a_3 + b_1b_3 + c_1c_3 = 0,$$

$$a_1a_2 + b_1b_2 + c_1c_2 = 0. \quad \text{(Vide Appendix.)}$$

Now

$$a_1\frac{da_1}{dt} + a_2\frac{da_2}{dt} + a_3\frac{da_3}{dt} = 0, \qquad (1)$$

$$b_1\frac{da_1}{dt} + b_2\frac{da_2}{dt} + b_3\frac{da_3}{dt} = -\omega_z, \qquad (2)$$

$$c_1\frac{da_1}{dt} + c_2\frac{da_2}{dt} + c_3\frac{da_3}{dt} = \omega_y. \qquad (3)$$

$$\therefore\ a_1(1) + b_1(2) + c_1(3) \text{ gives}$$

$$\frac{da_1}{dt} = c_1\omega_y - b_1\omega_z.$$

Similar values result for the differential coefficients of the remaining eight cosines.

65. We will now moreover shew that the motion of the body which we have proved to be one of rotation about an axis, may also be represented by three coexistent rotations about the co-ordinate axes.

Suppose the body to have angular velocities

$$\left.\begin{matrix}\omega_x\\ \omega_y\\ \omega_z\end{matrix}\right\} \text{ from } \left\{\begin{matrix}y\\ z\\ x\end{matrix}\right. \text{ towards } \left\{\begin{matrix}z\\ x\\ y\end{matrix}\right. \text{ about the axis of } \left\{\begin{matrix}x\\ y\\ z.\end{matrix}\right.$$

These angular velocities if they exist separately produce linear velocities in the particle x, y, z,

$$\left.\begin{matrix}y\omega_z\\ -z\omega_y\end{matrix}\right\} \text{ parallel to } \left\{\begin{matrix}z\\ y\end{matrix}\right.;$$

$$\left.\begin{matrix}z\omega_x\\ -x\omega_z\end{matrix}\right\} \text{ parallel to } \left\{\begin{matrix}x\\ z\end{matrix}\right.;$$

$$\left.\begin{matrix}x\omega_y\\ -y\omega_x\end{matrix}\right\} \text{ parallel to } \left\{\begin{matrix}y\\ x\end{matrix}\right. . \quad (40).$$

Hence if the three angular velocities be supposed to coexist without interference, the velocities of x, y, z are

$$\left.\begin{matrix}z\omega_y - y\omega_z\\ x\omega_z - z\omega_x\\ y\omega_x - x\omega_y\end{matrix}\right\} \text{ parallel to } \left\{\begin{matrix}x,\\ y,\\ z.\end{matrix}\right.$$

Hence it appears that when a body rotates about a fixed point, its motion may be truly represented by supposing three angular velocities about perpendicular axes to coexist. These angular velocities are known at each instant if the velocities at that instant of any particle of the body be given; and, conversely, if they be known, the motion of each particle of the body is determined.

66. Since at any instant the motion of the body is correctly exhibited by rotations properly chosen about three perpendicular axes, and since the position of these axes is arbitrarily adopted, the motion at different instants can also be represented by rotations about different systems of axes originating in the fixed point. Hence the body's motion may furthermore be exhibited by rotations about axes which themselves move after any assignable law, a statement which will include the case of axes so moving in space as to retain fixed positions in the body. If these moving axes were at any instant to become fixed, the angular velocities about them would by their coexistence exhibit the motion of the body at that instant.

67. *Composition and resolution of angular velocities.*

Since the motion of the body is exhibited either by a single angular velocity ω about an axis whose direction cosines are $\frac{\omega_x}{\omega}$, $\frac{\omega_y}{\omega}$, $\frac{\omega_z}{\omega}$ (63), or by three coexistent rotations ω_x, ω_y, ω_z about the co-ordinate axes (65), one of these modes of representing the same motion may be replaced by the other. Hence if a rigid body have an angular velocity ω about an axis through a fixed point whose direction cosines are α, β, γ, this velocity may be resolved into the coexistent angular velocities $\alpha\omega$, $\beta\omega$, $\gamma\omega$ about the axes of x, y, z respectively; or these coexistent velocities may be compounded into the single angular velocity ω.

68. To determine the motion of a body fixed at one point when two given angular velocities about given axes coexist in it.

Let ω, ω' be the given angular velocities about axes through the fixed point whose direction cosines are α, β, γ, and α', β', γ'.

These angular velocities are equivalent to

$$\left.\begin{matrix}\alpha\omega\\ \beta\omega\\ \gamma\omega\end{matrix}\right\} \text{ and } \left\{\begin{matrix}\alpha'\omega'\\ \beta'\omega'\\ \gamma'\omega'\end{matrix}\right\} \text{ about the axis of } \begin{cases}x,\\ y,\\ z;\end{cases}$$

and since they coexist, they are therefore equivalent to the single angular velocities

$$\left.\begin{matrix}\alpha\omega + \alpha'\omega' \\ \beta\omega + \beta'\omega' \\ \gamma\omega + \gamma'\omega'\end{matrix}\right\} \text{ about the axis of } \begin{cases}x, \\ y, \\ z.\end{cases}$$

These are equivalent to the angular velocity,

$$\sqrt{(\alpha\omega + \alpha'\omega')^2 + (\beta\omega + \beta'\omega')^2 + (\gamma\omega + \gamma'\omega')^2}.$$

$$= \sqrt{\omega^2 + \omega'^2 + 2\omega\omega'(\alpha\alpha' + \beta\beta' + \gamma\gamma')}, \qquad \text{(B)}$$

about the axis whose equations are

$$\frac{X}{\alpha\omega + \alpha'\omega'} = \frac{Y}{\beta\omega + \beta'\omega'} = \frac{Z}{\gamma\omega + \gamma'\omega'}.$$

69. The angular velocity of a rigid body about an axis may be properly represented by a straight line in direction of that axis. For the length of this line will represent the magnitude of the angular velocity according to some assumed standard, and the direction of the line will represent the direction of the angular velocity, if the convention be made that when the eye looks in the direction which is accounted positive in the measurement of the line, then the particles of a body having a positive angular velocity appear to rotate from right to left, or oppositely to the hands of a watch.

70. By help of this convention since $\alpha\alpha' + \beta\beta' + \gamma\gamma'$ is the cosine of the angle between the two given axes in (68), equation (B) indicates the following property :

If two straight lines drawn from a point represent in magnitude and direction two angular velocities coexistent in a rigid body of which that point is fixed, and if the parallelogram of which these lines are adjacent sides be completed, the resultant angular velocity of the body will be represented in quantity and direction by that diagonal of the parallelogram which is drawn from the intersection of the same two straight lines.

71. If there be more than two angular velocities coexistent in the body whereof one ω is about the axis whose direction cosines are α, β, γ, the resultant angular velocity

$$= \sqrt{\{\Sigma(\alpha\omega)\}^2 + \{\Sigma(\beta\omega)\}^2 + \{\Sigma(\gamma\omega)\}^2}$$

about the axis

$$\frac{X}{\Sigma(\alpha\omega)} = \frac{Y}{\Sigma(\beta\omega)} = \frac{Z}{\Sigma(\gamma\omega)}.$$

72. Let ABC be a triangle, in which C is a right angle.

Let AB represent the angular velocity of the body in magnitude and direction. If the parallelogram with sides AC, CB be completed, it is seen that the rotation AB is equivalent to rotations represented by AC, CB coexistent in the body. Now BC is perpendicular to AC; hence AC is the component of the body's angular velocity about an axis in this direction.

73. If a body has rotations ω_x, ω_y, ω_z about rectangular co-ordinate axes, its angular velocity about any axis whose direction cosines are α, β, γ is the sum of the components of those rotations about this axis, or $\alpha\omega_x + \beta\omega_y + \gamma\omega_z$.

The analogy of these propositions with the resolution of forces and of linear velocities will readily be traced.

74. The motion of a body turning about a fixed point may be expressed by coexistent angular motions about rectangular axes originating in that point, fixed relatively to the body and moving with it (66). When such angular velocities are known the velocity of every particle is determined, and its direction is also known with reference to the moveable axes. It remains to be shewn how from the same angular velocities the position of the body in space may be ascertained. The following method is taken from Prof. O'Brien's tract on Precession and Nutation.

75. Let a sphere be described with radius unity about the fixed point which is the origin of co-ordinates. Let the

fixed co-ordinate axes meet this sphere in the points X, Y, Z, and let three rectangular axes fixed in the body and moveable with it meet the surface in A, B, C (fig. 6). Let ω_1, ω_2, ω_3 be the given angular velocities of the body about the latter, by the supposed coexistence of which its motion is exhibited, (66). Let the great circles ZC, BA, produced if necessary, meet in E. Let $XZC = \psi$, $ZC = \theta$, $ECA = \phi$. If these angles be known the position of the body in space is completely assigned. (62).

Now if C be a point of the body, or be rigidly united with it, the velocity of $C = \begin{cases} \frac{d\theta}{dt} \text{ along } ZC, \\ \frac{d\psi}{dt} \sin\theta \text{ perpendicular to } ZC. \end{cases}$

But the velocity of C is likewise exhibited by ω_1 about A in direction BC, and ω_2 about B in direction CA.

$$\therefore \frac{d\theta}{dt} = \omega_1 \sin\phi + \omega_2 \cos\phi, \qquad (1)$$

$$\frac{d\psi}{dt} \sin\theta = -\omega_1 \cos\phi + \omega_2 \sin\phi. \qquad (2)$$

Again, the velocity of the point corresponding to E in direction AB is $\frac{d\phi}{dt} + \frac{d\psi}{dt} \cos\theta$;

$$\therefore \frac{d\phi}{dt} + \frac{d\psi}{dt} \cos\theta = \omega_3. \qquad (3)$$

If then ω_1, ω_2, ω_3 be given these equations after elimination and integration give θ, ϕ, ψ, and determine the position of the body. The constants which integration introduces are to be assigned from the given position of the body at some given epoch.

76. Hence the angular velocities of the body about a system of rectangular axes fixed in itself may be considered the elements of its motion, from which the position, velocity and direction of any point of it at a given instant can be known.

Equations of motion.

77. The preceding propositions are designed to give a conception of the geometrical nature of a body's motion about a fixed point. We have now to compute its motion from a knowledge of the forces which cause it. The properties of principal axes make it most convenient to investigate the angular velocities about such a system, and to exhibit the body's motion by these angular velocities in the sense explained in (66).

I. *Finite forces.*

78. When a rigid body is in motion about a fixed point under given finite forces, to find its angular velocities about a system of principal axes through that point.

Let the fixed point be made origin, and let x, y, z be the co-ordinates at time t of a particle of the body of mass m, referred to fixed rectangular axes, X, Y, Z the given impressed accelerating forces on this particle parallel to these axes.

Then if the effective moving forces, represented generally by $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$, $m\frac{d^2z}{dt^2}$, be applied to the particles to which they belong in directions contrary to their own, the conditions of equilibrium will be satisfied among these and the impressed forces (30);

$$\therefore\ \Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = \Sigma m(Zy - Yz) = L,$$

$$\Sigma m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) = \Sigma m(Xz - Zx) = M,$$

$$\Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = \Sigma m(Yx - Xy) = N,$$

L, M, N being known from the given forces applied to the body, the unknown reaction of the fixed point having no moment about the co-ordinate axes.

Let ω_x, ω_y, ω_z be the coexistent rotations about the coordinate axes of x, y, z which can represent the body's motion (65);

$$\therefore \frac{dx}{dt} = z\omega_y - y\omega_z,$$

$$\frac{dy}{dt} = x\omega_z - z\omega_x,$$

$$\frac{dz}{dt} = y\omega_x - x\omega_y.$$

$$\therefore \frac{d^2z}{dt^2} = y \cdot \frac{d\omega_x}{dt} - x\frac{d\omega_y}{dt} + \omega_x(x\omega_z - z\omega_x) - \omega_y(z\omega_y - y\omega_z),$$

$$\frac{d^2y}{dt^2} = x\frac{d\omega_z}{dt} - z\frac{d\omega_x}{dt} + \omega_z(z\omega_y - y\omega_z) - \omega_x(y\omega_x - x\omega_y);$$

$$\therefore \Sigma m(y^2 + z^2) \cdot \frac{d\omega_x}{dt} + \Sigma m(y^2 - z^2) \cdot \omega_y\omega_x$$

$$- \Sigma(mxy) \cdot \frac{d\omega_y}{dt} - \Sigma(mxz) \cdot \frac{d\omega_z}{dt}$$

$$+ \Sigma(mxy) \cdot \omega_x\omega_z - \Sigma(mxz) \cdot \omega_x\omega_y$$

$$- \Sigma(myz) \cdot \omega_x^2 + \Sigma(myz) \cdot \omega_z^2$$

$$- \Sigma(myz)\omega_y^2 + \Sigma(myz)\omega_x^2 = L.$$

The other two equations of moments may be similarly transformed.

Let ω_1, ω_2, ω_3 be the angular velocities about a system of principal axes through the origin which may likewise represent the motion of the body at time t. (66).

If a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, be the direction cosines of these axes with reference to the fixed axes,

$$\omega_x = a_1\omega_1 + a_2\omega_2 + a_3\omega_3.$$

$$\therefore \quad \frac{d\omega_x}{dt} = a_1\frac{d\omega_1}{dt} + a_2\frac{d\omega_2}{dt} + a_3\frac{d\omega_3}{dt}$$

$$+ (a_1\omega_x + b_1\omega_y + c_1\omega_z)\frac{da_1}{dt}$$

$$+ (a_2\omega_x + b_2\omega_y + c_2\omega_z)\frac{da_2}{dt}$$

$$+ (a_3\omega_x + b_3\omega_y + c_3\omega_z)\frac{da_3}{dt}.$$

Here the coefficient of $\omega_x = a_1\frac{da_1}{dt} + a_2\frac{da_2}{dt} + a_3\frac{da_3}{dt} = 0,$

..... $\omega_y = b_1\frac{a\,d_1}{dt} + b_2\frac{da_2}{dt} + b_3\frac{da_3}{dt} = -\omega_z,$

..... $\omega_z = c_1\frac{da_1}{dt} + c_2\frac{da_2}{dt} + c_3\frac{da_3}{dt} = \omega_y.$

$$\therefore \quad \frac{d\omega_x}{dt} = a_1\frac{d\omega_1}{dt} + a_2\frac{d\omega_2}{dt} + a_3\frac{d\omega_3}{dt}.$$

So $$\frac{d\omega_y}{dt} = b_1\frac{d\omega_1}{dt} + b_2\frac{d\omega_2}{dt} + b_3\frac{d\omega_3}{dt},$$

$$\frac{d\omega_z}{dt} = c_1\frac{d\omega_1}{dt} + c_2\frac{d\omega_2}{dt} + c_3\frac{d\omega_3}{dt}.$$

Now let the axes of x, y, z have been so assumed that at the time t the principal axes coincide with them.

Then $\Sigma(myz) = 0, \quad \Sigma(mxz) = 0, \quad \Sigma(mxy) = 0,$

$$\omega_x = \omega_1, \qquad \omega_y = \omega_2, \qquad \omega_z = \omega_3,$$

and since $a_1=b_2=c_3=1,\ a_2=0,\ a_3=0,\ b_1=0,\ b_3=0,\ c_1=0,\ c_2=0,$

$$\therefore \quad \frac{d\omega_x}{dt} = \frac{d\omega_1}{dt}, \qquad \frac{d\omega_y}{dt} = \frac{d\omega_2}{dt}, \qquad \frac{d\omega_z}{dt} = \frac{d\omega_3}{dt}.$$

Hence if A, B, C be the moments of inertia of the body about the principal axes to which ω_1, ω_2, ω_3 respectively refer,

$$A\,d_t\omega_1 + (C - B)\,\omega_2\,\omega_3 = L,$$

$$B\,d_t\omega_2 + (A - C)\,\omega_1\,\omega_3 = M,$$

$$C\,d_t\omega_3 + (B - A)\,\omega_1\,\omega_2 = N.$$

These equations do not recognize the axes of x, y, z, and therefore are not restricted by the particular position in which those axes were for convenience assumed. They are therefore the three equations for determining the body's angular velocities about the given principal axes, about which L, M, N are the moments of the impressed forces.

79. When ω_1, ω_2, ω_3 are thus determined, the body's motion is represented by a single angular velocity $\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$ about a straight line whose equations referred to the principal axes are $\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3}$. The position of the body in space will be known from the equations of (75).

80. To find the pressure on the fixed point.

If $\Sigma(mX)$, $\Sigma(mY)$, $\Sigma(mZ)$ be the sum of the components of the given forces resolved parallel to the co-ordinate axes, F, G, H the pressures in the same directions on the fixed point,

$$\Sigma(mX) - F = \Sigma\left(\frac{m\,d^2x}{dt^2}\right),$$

$$\Sigma(mY) - G = \Sigma\left(\frac{m\,d^2y}{dt^2}\right),$$

$$\Sigma(mZ) - H = \Sigma\left(\frac{m\,d^2z}{dt^2}\right),$$

whereby, if the values of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ in terms of the angular velocities be substituted (78), the pressures on the point are known.

If a body turns about its centre of gravity, the pressures on that point in direction of the co-ordinate axes are

$$\Sigma(mX),\quad \Sigma(mY),\quad \Sigma(mZ).$$

81. In the case where no force acts on the rigid body except at the fixed point, the equations of motion lead to some remarkable results which shall now be exhibited.

If no forces act on the body which would tend to produce motion in it if it were at rest, $L = 0$, $M = 0$, $N = 0$;

$$\therefore\ A\frac{d\omega_1}{dt} + (C - B)\,\omega_2\,\omega_3 = 0, \qquad (1)$$

$$B\frac{d\omega_2}{dt} + (A - C)\,\omega_1\,\omega_3 = 0, \qquad (2)$$

$$C\frac{d\omega_3}{dt} + (B - A)\,\omega_1\,\omega_2 = 0; \qquad (3)$$

Hence $\omega_1(1) + \omega_2(2) + \omega_3(3)$,

and $A\omega_1 . (1) + B\omega_2 . (2) + C\omega_3 . (3)$, give

$$A\omega_1\frac{d\omega_1}{dt} + B\omega_2\frac{d\omega_2}{dt} + C\omega_3\frac{d\omega_3}{dt} = 0,$$

$$A^2\omega_1\frac{d\omega_1}{dt} + B^2\omega_2\frac{d\omega_2}{dt} + C^2\omega_3\frac{d\omega_3}{dt} = 0;$$

whence by integration,

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = h^2,$$

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = k^2;$$

the latter members of these equations being positive quantities depending on the primitive circumstances of the motion.

82. Also when the body is referred to axes fixed in space,

$$\Sigma m\left(y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2}\right) = 0,$$

$$\therefore\ \Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) = h_1\ .$$

$$\text{So}\quad \Sigma m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) = h_2,$$

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = h_3,$$

h_1, h_2, h_3 being constants dependent on the primitive circumstances of motion.

Let another system of fixed rectangular axes from the same origin be adopted, and let a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, be their direction cosines in reference to the former system, x', y', z' the co-ordinates of m referred to them. Since

$$\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)\delta t, \qquad \left(y'\frac{dz'}{dt} - z'\frac{dy'}{dt}\right)\delta t, \ \ldots$$

are ultimately doubles of the sectorial areas described in time δt by the projection of m upon the planes of yz, $y'z'$, ...

$$\therefore\ h_1 = \Sigma m\left(y'\frac{dz'}{dt} - z'\frac{dy'}{dt}\right)a_1 + \Sigma m\left(z'\frac{dx'}{dt} - x'\frac{dz'}{dt}\right)a_2 + \Sigma m\left(x'\frac{dy'}{dt} - y'\frac{dx'}{dt}\right)a_3.$$

$$\text{Now}\quad \Sigma m\left(y'\frac{dz'}{dt} - z'\frac{dy'}{dt}\right) = \Sigma m(y'^2 + z'^2)\omega_{x'} - \Sigma(mx'y')\omega_{y'} - \Sigma(mx'z')\omega_{z'},$$

and the other two functions may be similarly transformed.

If then the second system of co-ordinate axes be so chosen as to coincide at the instant considered with principal axes of the body,

$$h_1 = A\omega_1 a_1 + B\omega_2 a_2 + C\omega_3 a_3.$$

$$\text{So}\quad h_2 = A\omega_1 b_1 + B\omega_2 b_2 + C\omega_3 b_3,$$

$$h_3 = A\omega_1 c_1 + B\omega_2 c_2 + C\omega_3 c_3.$$

83. Hence $h_1^2 + h_2^2 + h_3^2 = k^2$.

84. Also $h_1a_1 + h_2b_1 + h_3c_1 = A\omega_1$,

$$h_1a_2 + h_2b_2 + h_3c_2 = B\omega_2,$$

$$h_1a_3 + h_2b_3 + h_3c_3 = C\omega_3.$$

85. If H be the double of the velocity with which sectorial areas are described by the molecule m on any plane whose direction cosines are α, β, γ,

$$\Sigma(mH) = h_1\alpha + h_2\beta + h_3\gamma.$$

$\Sigma(mH)$ will be greatest when $\frac{\alpha}{h_1} = \frac{\beta}{h_2} = \frac{\gamma}{h_3} = \frac{1}{\sqrt{h_1^2 + h_2^2 + h_3^2}}$.

86. This plane, whose equation is $0 = h_1x + h_2y + h_3z$, is denominated the invariable plane of the body, on account of properties which it will hereafter be seen to possess. If we denote its normal by I, whose direction cosines are $\frac{h_1}{k}$, $\frac{h_2}{k}$, $\frac{h_3}{k}$, and call a the principal axis to which A belongs, we have

$$\cos(I, a) = \frac{h_1a_1 + h_2b_1 + h_3c_1}{k} = \frac{A\omega_1}{k}. \quad (84).$$

Also with the notation and figure of (75), by supposing the invariable plane to coincide with that of xy, and the point A to belong to the principal axis a,

$$\cos(I, a) = \cos ZA = -\sin\theta\cos\phi;$$

$$\therefore A\omega_1 = -k\sin\theta\cos\phi.$$

$$\text{So } B\omega_2 = k\sin\theta\sin\phi,$$

$$\text{and } C\omega_3 = k\cos\theta.$$

$$\text{Hence } \frac{d\theta}{dt} = -k\sin\theta\,.\sin\phi\cos\phi\left(\frac{1}{A} - \frac{1}{B}\right).$$

$$\frac{d\psi}{dt}\sin\theta = k\sin\theta\left(\frac{\cos^2\phi}{A} + \frac{\sin^2\phi}{B}\right),$$

$$\frac{d\phi}{dt} + \frac{d\psi}{dt}\cos\theta = \frac{k\cos\theta}{C}. \quad (75.)$$

87. When the principal axes are axes of co-ordinates, the equation to the central ellipsoid is

$$Ax^2 + By^2 + Cz^2 = 1. \quad (22).$$

The instantaneous axis referred to the same axes having $\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3}$ for its equations, meets this ellipsoid in a point whose co-ordinates are $\frac{\omega_1}{h}$, $\frac{\omega_2}{h}$, $\frac{\omega_3}{h}$.

The tangent plane to the ellipsoid at this point is

$$A\omega_1 x + B\omega_2 y + C\omega_3 z = h,$$

and when axes fixed in space are adopted as co-ordinate axes the equation to this tangent plane becomes

$$A\omega_1(a_1x+b_1y+c_1z)+B\omega_2(a_2x+b_2y+c_2z)+C\omega_3(a_3x+b_3y+c_3z)=h,$$

$$\text{or } h_1x + h_2y + h_3z = h. \quad (82).$$

The tangent plane to the central ellipsoid at the point where the instantaneous axis meets it is therefore fixed in space during the motion.

88. The motion of the central ellipsoid therefore is under these two conditions: (1) its centre is fixed, (2) it always touches a fixed plane. Since also the point of contact lies in the instantaneous axis, that point has no velocity. Hence the motion of the central ellipsoid is such that while its centre is fixed it rolls upon a fixed plane whose distance from its centre is $\frac{h}{k}$.

The plane $h_1x + h_2y + h_3z = 0$ is from this property termed the invariable plane of the rigid body. (86).

89. When the body is a solid of revolution turning about a point in its axis, the central ellipsoid becomes a spheroid whose axis is that of the body, and the instantaneous axis

describes a circular right cone whose base is parallel to the invariable plane.

90. If the central ellipsoid becomes a sphere in consequence of the principal moments of inertia of the body at the point about which it turns being equal, then the instantaneous axis remains fixed in space as well as in the body.

91. The equation to the invariable plane referred to axes fixed in space and so taken that the principal axes of the body coincided with them at a given epoch is

$$0 = A\Omega_1 x + B\Omega_2 y + C\Omega_3 z,$$

where Ω_1, Ω_2, Ω_3 are the values of ω_1, ω_2, ω_3 at that instant.

92. If the instantaneous axis $\frac{x}{\Omega_1} = \frac{y}{\Omega_2} = \frac{z}{\Omega_3}$ lie in the invariable plane, $A\Omega_1^2 + B\Omega_2^2 + C\Omega_3^2 = 0$, which is impossible since A, B, C are positive quantities.

The instantaneous axis will be perpendicular to the invariable plane if the straight lines

$$\frac{x}{\Omega_1} = \frac{y}{\Omega_2} = \frac{z}{\Omega_3}, \text{ and } \frac{x}{A\Omega_1} = \frac{y}{B\Omega_2} = \frac{z}{C\Omega_3}$$

coincide.

This will happen (1) if $A = B = C$;

(2) if two of the quantities Ω_1, Ω_2, Ω_3 vanish together.

93. The radius vector of the central ellipsoid drawn in direction of the instantaneous axis,

$$= \frac{1}{h}\sqrt{(\omega_1^2 + \omega_2^2 + \omega_3^2)},$$

and is therefore proportional to the angular velocity of the body (63, 66).

94. The following is an additional illustration of the nature of the body's motion.

Since
$$\left.\begin{aligned} A\omega_1^2 + B\omega_2^2 + C\omega_3^2 &= h^2 \\ A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 &= k^2 \end{aligned}\right\};$$

$$\therefore\ A\left(\frac{A}{k^2} - \frac{1}{h^2}\right)\omega_1^2 + B\left(\frac{B}{k^2} - \frac{1}{h^2}\right)\omega_2^2 + C\left(\frac{C}{k^2} - \frac{1}{h^2}\right)\omega_3^2 = 0.$$

Now the equations to the instantaneous axis referred to the principal axes of the body are

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3};$$

$$\therefore\ A\left(\frac{A}{k^2} - \frac{1}{h^2}\right)x^2 + B\left(\frac{B}{k^2} - \frac{1}{h^2}\right)y^2 + C\left(\frac{C}{k^2} - \frac{1}{h^2}\right)z^2 = 0, \qquad (a).$$

Hence the instantaneous axis describes in the body a conical surface of the second order. It likewise describes in space a fixed conical surface, and these two cones have always a common generatrix, viz. the instantaneous axis, and every point of this instantaneous axis if it be rigidly united to the body has no velocity. Hence if we suppose the conical surface whose equation is (a) to roll on the fixed conical surface which is the locus of the axis in space, the motion of the body will be exhibited by supposing it rigidly attached to and carried along with the former rolling cone.

This geometrical illustration of rotatory motion about a point is due to Poinsot, and exhibits the double motion of the instantaneous axis in space and in the body. In space that axis describes a fixed conical surface; in the body it describes a cone of the second order which may in particular cases degenerate into a plane or a line when one or two of the coefficients in (a) vanish.

95. The motion of the Earth as a spheroid in consequence of Precession, which is one of the most important applications of the equations of this chapter, may be thus conceived.

Describe a circular cone whose vertex is at the centre of the spheroid, and its axis perpendicular to the ecliptic, and also another circular cone about the axis of the spheroid with the same vertex. Let the difference of their semi-vertical angles be the obliquity of the ecliptic, and the ratio of the semi-vertical angle of the latter to that of the former the ratio of a day to the period of the Precession of the Equinoxes. Then if the latter cone roll within the former fixed, with the Earth's mean angular velocity, it may complete its circuit in the period of the Equinoxes and revolve in space in one day, and therefore if the Earth be rigidly attached to it and carried with it, the motion of the Earth about its own centre in consequence of Precession is correctly exhibited.

96. The body, as has been seen (88), will permanently rotate about any of its principal axes at the fixed point about which it turns, but the three principal axes which form a system have different characters with respect to the stability of the rotation about them.

The path of the pole of the instantaneous axis on the surface of the central ellipsoid is given by the equations

$$Ax^2 + By^2 + Cz^2 = 1,$$

$$A^2x^2 + B^2y^2 + C^2z^2 = \frac{k^2}{h^2}.$$

Its projections therefore on the principal planes are

$$B(A - B)y^2 + C(A - C)z^2 = \left(A - \frac{k^2}{h^2}\right),$$

$$A(B - A)x^2 + C(B - C)z^2 = \left(B - \frac{k^2}{h^2}\right),$$

$$A(C - A)x^2 + B(C - B)y^2 = \left(C - \frac{k^2}{h^2}\right).$$

Let A, B, C be in descending order of magnitude.

(1) The projection of the pole's path on the plane yz is an ellipse, whose eccentricity $= \sqrt{\frac{B-C}{A-C}\cdot\frac{B+C-A}{C}}$. If this eccentricity is small, the position of the instantaneous axis will never deviate far from that of the principal axis of maximum moment if it is once near to it.

(2) The projection of the path on the plane xy is an ellipse whose eccentricity $= \sqrt{\frac{A-B}{A-C}\cdot\frac{A+B-C}{A}}$. If this eccentricity be small, a similar conclusion may be drawn respecting the axis of minimum moment.

Hence, under the conditions prescribed imposed on the eccentricities of these ellipses, rotation about the axis of maximum or minimum moment is said to be stable; for if the body be rotating about either and be slightly disturbed so that the position of the instantaneous axis is thrown slightly out of coincidence with either, it will still never deviate far from that with which it previously coincided.

The degree of stability about the two axes will be compared by means of the ratio of the eccentricities of the two ellipses which the pole's projections describe, or by

$$\sqrt{\frac{A}{C}\cdot\frac{B-C}{A-B}\cdot\frac{B+C-A}{A+B-C}}.$$

(3) The projection of the path on the plane xz is an hyperbola, and therefore the projection of the pole may depart widely from the centre, if it was originally near to it, so that rotation about the axis of mean moment will not generally be stable.

97. In one particular case however rotation about the mean axis may be stable.

Let $k^2 = Bh^2$,

$$\therefore\ A(A-B)\omega_1^2 = C(B-C)\omega_3^2,$$

$$AB(A-C)\omega_1^2 = (B-C)\{k^2 - B^2\omega_2^2\}, \qquad (1)$$

$$BC(A-C)\omega_3^2 = (A-B)\{k^2 - B^2\omega_2^2\}.$$

$$B^2 \frac{d\omega_2}{dt} = -B(A-C)\omega_1\omega_3$$

$$= \mp \sqrt{\frac{(A-B)(B-C)}{AC}}\,(k^2 - B^2\omega_2^2),$$

the upper or lower sign being used as ω_1 and ω_3 have like or unlike signs. Since by virtue of (1) $k^2 > B^2\omega_2^2$; $\therefore$ ω_2 continuously decreases or increases with the time in these cases respectively.

Now $$\frac{B}{k^2 - B^2\omega_2^2} \cdot \frac{d\omega_2}{dt} = \mp \frac{1}{B}\sqrt{\frac{(A-B)(B-C)}{AC}} = \mp n \text{ suppose.}$$

$$\therefore \frac{k + B\omega_2}{k - B\omega_2} = C' \epsilon^{\mp 2knt},$$

C' being an arbitrary constant.

Hence as t is indefinitely increased ω_2 approaches to $\mp \frac{k}{B}$ as its limit, and ω_1, ω_3 to zero.

Hence if $k^2 = Bh^2$, the mean axis is an axis of stable rotation, in this sense that if a body revolving about it be disturbed into revolving about another axis so that the condition aforesaid is preserved, the instantaneous axis will continually tend to restore itself into coincidence either with its original position or that position reversed by being produced backwards, and the cases are distinguished by the circumstance of ω_1 and ω_3 having at the beginning of the disturbance unlike or like signs respectively.

In this case the instantaneous axis constantly lies in one of the planes $\frac{x}{\sqrt{C(B-C)}} = \pm \frac{z}{\sqrt{A(A-B)}}$. Each of these planes cuts the central ellipsoid in a curve such that the perpendicular from the centre on the tangent plane at any point of the curve has the same length $B^{-\frac{1}{2}}$

98. In reviewing then the results of this section it appears that

(1) When a rigid body revolves under any circumstances about a fixed point, there is at every instant a line of particles at rest, and the motion of the body is at the instant identical with that of a body revolving about this line as a fixed axis. (63). The instantaneous axis is in general variable in position and traces out a conical surface of some species with the fixed point as vertex.

(2) When the forces acting on the rigid body have no moment about any straight line through the fixed point, then the motion is such that the central ellipsoid rolls upon a fixed tangent plane (88), the instantaneous axis being the diameter at the point of contact (88) and the angular velocity being proportional to the length of this diameter (93).

When the motion of the instantaneous axis in the body is considered, *i.e.* its motion as it would appear to an observer unconscious of the body's motion and employing for reference lines fixed in it as if they were fixed in space, then in this view the instantaneous axis describes a conical surface of some species about the fixed point as vertex, and in the case of the impressed forces having no moment about the fixed point, this cone is one of the second order (94).

II. *Impulsive forces.*

99. To determine the change of motion under given impulses of a body of which one point is fixed.

The fixed point being origin let x, y, z be the rectangular co-ordinates of a particle whose mass is m at the moment when the impulses act. Let ω_x, ω_y, ω_z be angular velocities about the axes of x, y, z which represent the body's motion before the impulses, and let them be suddenly changed into ω_x', ω_y', ω_z'.

Now the effective impulsive moving forces on m are

$$m\{(\omega_y' - \omega_y)z - (\omega_z' - \omega_z)y\} \text{ in direction of } x,$$

$$m\{(\omega_z' - \omega_z)x - (\omega_x' - \omega_x)z\} \quad \ldots\ldots\ldots\ldots \quad y,$$

$$m\{(\omega_x' - \omega_x)y - (\omega_y' - \omega_y)x\} \quad \ldots\ldots\ldots\ldots \quad z. \quad (28.\ 63.)$$

And between these in reversed directions and the impressed impulses the conditions of statical equilibrium are satisfied.

If then L, M, N be the moments of the given impulses about the axes, since the unknown reaction of the fixed point has no moment about these axes,

$$L = \Sigma[my\{(\omega_z' - \omega_z)y - (\omega_y' - \omega_y)x\}$$
$$- mz\{(\omega_z - \omega_z')x - (\omega_x' - \omega_x)z\}],$$
$$= \Sigma m(y^2+z^2).(\omega_x' - \omega_x) - \Sigma(myx).(\omega_y' - \omega_y) - \Sigma(mxz).(\omega_z' - \omega_z),$$

or if A, B, C be the moments of inertia of the body about the axes of x, y, z;

$$A' = \Sigma(myz), \quad B' = \Sigma(mxz), \quad C' = \Sigma(mxy),$$

then

$$L = A(\omega_x' - \omega_x) - C'(\omega_y' - \omega_y) - B'(\omega_z' - \omega_z),$$

$$M = B(\omega_y - \omega_y) - C'(\omega_x' - \omega_x) - A'(\omega_z' - \omega_z),$$

$$N = C(\omega_z' - \omega_z) - B'(\omega_x' - \omega_x) - A'(\omega_y' - \omega_y).$$

When ω_x', ω_y', ω_z' are obtained from these equations, the initial motion of the body after the impulses is expressed by an angular velocity $\sqrt{\omega_x'^2 + \omega_y'^2 + \omega_z'^2}$ about the axis

$$\frac{x'}{\omega_x'} = \frac{y'}{\omega_y'} = \frac{z'}{\omega_z'}.$$

Hence the velocity and direction of every point of the body are known.

100. If the axes of co-ordinates be so selected as to coincide with principal axes of the body at the origin,

$$A(\omega_x' - \omega_x) = L, \quad B(\omega_y' - \omega_y) = M, \quad C(\omega_z' - \omega_z) = N.$$

101. To find the impulses on the fixed point.

Let F, G, H be the impulses on the fixed point, and these in reversed directions the reactions on the body. Then if $\Sigma(X)$, $\Sigma(Y)$, $\Sigma(Z)$ are the algebraic sums of the given impulses parallel to the co-ordinate axes, $\bar{x}$, $\bar{y}$, $\bar{z}$ co-ordinates of the centre of gravity of the body,

$$\begin{aligned}
\Sigma(X) - F &= \Sigma\{mz(\omega_y' - \omega_y) - my(\omega_z' - \omega_z)\} \\
&= \{(\omega_y' - \omega_y)\bar{z} - (\omega_z' - \omega_z)\bar{y}\} \cdot \Sigma(m), \\
\Sigma(Y) - G &= \{(\omega_z' - \omega_z)\bar{x} - (\omega_x' - \omega_x)\bar{z}\} \cdot \Sigma(m), \\
\Sigma(Z) - H &= \{(\omega_x' - \omega_x)\bar{y} - (\omega_y' - \omega_y)\bar{x}\} \cdot \Sigma(m);
\end{aligned}$$

whereby the impulses on the fixed point are known, ω_x', ω_y', ω_z' being already determined. (99).

If the axes be principal axes,

$$F = \Sigma(X) - \left\{\frac{M\bar{z}}{B} - \frac{N\bar{y}}{C}\right\}\Sigma(m),$$

$$G = \Sigma(Y) - \left\{\frac{N\bar{x}}{C} - \frac{L\bar{z}}{A}\right\}\Sigma(m),$$

$$H = \Sigma(Z) - \left\{\frac{L\bar{y}}{A} - \frac{M\bar{x}}{B}\right\}\Sigma(m).$$

102. If the axes be principal axes, the equations to the initial instantaneous axis of a body previously at rest are

$$\frac{Ax}{L} = \frac{By}{M} = \frac{Cz}{N}.$$

The forces acting on the body consist of a force at the origin and a couple whose plane is

$$Lx + My + Nz = 0.$$

Hence if the central ellipsoid be described whose equation is

$$Ax^2 + By^2 + Cz^2 = 1,$$

the plane of the couple is diametral in this or any similar, concentric, and similarly situated ellipsoid to the instantaneous axis.

The instantaneous axis is thus never perpendicular to the plane of the couple except when that plane is perpendicular to a principal axis of the body at the fixed point.

SECTION V.

MOTION OF A FREE RIGID BODY.

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103. Geometrical conception of the motion.

The most general motion of an unconstrained rigid body may be represented by supposing two motions coexisting;

(1) A motion of translation, whereby each particle has the velocity in magnitude and direction of an assumed point which either belongs to the body or is in imaginary rigid union with it, and

(2) A motion of rotation about an axis through the said assumed point, whereby each particle has the velocity in magnitude and direction arising from rotation about this axis.

The possibility of correctly exhibiting a body's motion by this means appears from the following proof.

Let x, y, z be rectangular co-ordinates at time t of a particle of the body referred to axes fixed in space, α, β, γ co-ordinates of another point which is made the origin of a system of rectangular co-ordinates moveable with the body. Let x', y', z' be co-ordinates referred to these latter.

Then
$$x' = a_1(x-\alpha) + b_1(y-\beta) + c_1(z-\gamma),$$
$$y' = a_2(x-\alpha) + b_2(y-\beta) + c_2(z-\gamma),$$
$$z' = a_3(x-\alpha) + b_3(y-\beta) + c_3(z-\gamma);$$

where a_1, b_1, ... depend on the position of the whole body and not on any particular particle of it, and are connected by the conditions

$$a_1^2 + a_2^2 + a_3^2 = b_1^2 + b_2^2 + b_3^2 = c_1^2 + c_2^2 + c_3^2 = 1,$$
$$a_1b_1 + a_2b_2 + a_3b_3 = 0, \quad a_1c_1 + a_2c_2 + a_3c_3 = 0, \qquad (A)$$
$$b_1c_1 + b_2c_2 + b_3c_3 = 0.$$

$$\therefore\ 0 = a_1\frac{d(x-\alpha)}{dt} + b_1\frac{d(y-\beta)}{dt} + c_1\frac{d(z-\gamma)}{dt}$$

$$+ (x-\alpha)\frac{da_1}{dt} + (y-\beta)\frac{db_1}{dt} + (z-\gamma)\frac{dc_1}{dt} \quad (1)$$

$$0 = a_2\frac{d(x-\alpha)}{dt} + b_2\frac{d(y-\beta)}{dt} + c_2\frac{d(z-\gamma)}{dt}$$

$$+ (x-\alpha)\frac{da_2}{dt} + (y-\beta)\frac{db_2}{dt} + (z-\gamma)\frac{dc_2}{dt} \quad (2)$$

$$0 = a_3\frac{d(x-\alpha)}{dt} + b_3\frac{d(y-\beta)}{dt} + c_3\frac{d(z-\gamma)}{dt}$$

$$+ (x-\alpha)\frac{da_3}{dt} + (y-\beta)\frac{db_3}{dt} + (z-\gamma)\frac{dc_3}{dt} \quad (3).$$

$a_1(1) + a_2(2) + a_3(3)$ gives by virtue of (A),

$$0 = \frac{d(x-\alpha)}{dt} + (y-\beta)\left(a_1\frac{db_1}{dt} + a_2\frac{db_2}{dt} + a_3\frac{db_3}{dt}\right)$$

$$+ (z-\gamma)\left(a_1\frac{dc_1}{dt} + a_2\frac{dc_2}{dt} + a_3\frac{dc_3}{dt}\right),$$

$$\text{or } \frac{dx}{dt} = \frac{d\alpha}{dt} + (z-\gamma)\omega_y - (y-\beta)\omega_z, \text{ suppose,}$$

$$\text{so } \frac{dy}{dt} = \frac{d\beta}{dt} + (x-\alpha)\omega_z - (z-\gamma)\omega_x,$$

$$\frac{dz}{dt} = \frac{d\gamma}{dt} + (y-\beta)\omega_x - (x-\alpha)\omega_y;$$

wherein ω_x, ω_y, ω_z are dependent on the position of the body in space.

Now in each of these expressions the first terms are the component velocities of the assumed point α, β, γ; on sup-

position of its rigid union with the rest of the body; the latter two terms are the velocities which would result from the body having an angular velocity $\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ about an axis through α, β, γ parallel to the straight line $\frac{X}{\omega_x} = \frac{Y}{\omega_y} = \frac{Z}{\omega_z}$.

Therefore the motion of the body is such as would result from the co-existence of the motion of all its particles in parallel directions and the motion of rotation aforesaid. The former motion is called the motion of translation, and then the proposition may be stated that the motion of a rigid body may be always supposed to result from the coexistence of a motion of translation and a motion of rotation. These will be the elements of the body's motion from a determination of which the place, velocity, and direction of every point of it will be known.

104. For reasons which will appear hereafter it will be usually expedient to allow the assumed point, mentioned in the preceding article as determining by its motion the motion of translation of the body, to be the centre of gravity of the body.

105. If a rigid body has two coexistent angular velocities about parallel axes, to find the nature of its motion.

Let the rigid body have angular velocites ω, ω' in the same direction about axes whose traces on a plane perpendicular to each are A, B (fig. 7.) In AB produced take any point P. Then from these angular velocities separately the velocities of P regarded as a point of the body or rigidly united with it, would be $\omega \,.\, AP$ and $\omega' \,.\, BP$ in the same direction in the plane on which A, B are traces. Hence if the two angular velocities coexist the velocity of P is

$$\omega \,.\, AP + \omega' \,.\, BP.$$

The point P has no velocity if

$$\omega \,.\, AP + \omega' \,.\, BP = 0,$$

$$\text{or } AP = \frac{\omega' \,.\, AB}{\omega + \omega'};$$

and the same will be true of every point in the straight line through P perpendicular to the assumed plane; therefore the body will rotate about this line.

If ω_1 be the angular velocity about the axis through P, then from the two expressions for the velocity of A,

$$\omega_1 . AP = \omega' . AB,$$

$$\therefore \omega_1 = \omega + \omega'.$$

106. If $\omega = -\omega'$, these results are nugatory. In this case the velocity of any point P in AB or AB produced

$$= \omega . AP + \omega' . BP = \omega . AB.$$

All points therefore in the plane of the two axes A, B have the same velocity in a direction perpendicular to that plane. The motion of the body is therefore a motion of translation.

DEF. Equal angular velocities existing in opposite directions in a rigid body form a couple of rotatory motion.

Hence a velocity of translation may be replaced by a pair of equal and opposite angular velocities about axes which are parallel to one another and perpendicular to the direction of that velocity, the distance of the axes being the velocity of translation divided by the angular velocity.

107. Hence the motion of a body can always be reduced to two rotations at most about two axes. For after its motion has been reduced to a translation and a rotation, the former may be replaced by rotations about two axes, of which one meets the axis of rotation, and the three rotations may then be compounded into two at most. (103, 106, 68.)

108. Couples of rotatory motion are equivalent if the planes through the axes are parallel and the product of the distance of the axes and the angular velocity about each be the same. A couple of rotatory motion may be resolved in any

plane different from its own by multiplying its magnitude by the cosine of the angle between that plane and its own.

109. When the motion of a rigid body referred to rectangular co-ordinates is expressed by the velocities u, v, w of the origin of co-ordinates, which is either actually or hypothetically in rigid union with the body, together with angular velocities ω_x, ω_y, ω_z about the same axes, these angular velocities are the same whatever origin be adopted.

For let u', v', w' be the linear velocities of a point α, β, γ, ω_x', ω_y', ω_z' the angular velocities about it when it is made origin. Then expressing the velocities of a particle x, y, z on each supposition we have

$$\left.\begin{aligned} u + z\omega_y - y\omega_z &= u' + (z-\gamma)\omega_y' - (y-\beta)\omega_z' \\ v + x\omega_z - z\omega_x &= v' + (x-\alpha)\omega_z' - (z-\gamma)\omega_x' \\ w + y\omega_x - x\omega_y &= w' + (y-\beta)\omega_x' - (x-\alpha)\omega_y' \end{aligned}\right\}.$$

These equations hold for all values of x, y, and z.

$$\therefore \frac{\omega_x - \omega_x'}{x} = \frac{\omega_y - \omega_y'}{y} = \frac{\omega_z - \omega_z'}{z}.$$

Since the angular velocities are independent of x, y, z and the latter are unconnected,

$$\therefore \omega_x = \omega_x', \quad \omega_y = \omega_y', \quad \omega_z = \omega_z'.$$

Thus whatever origin be adopted the motion of rotation whereby the body's motion is exhibited is the same, while the motion of translation in general varies.

110. Since
$$u' = u + \gamma\omega_y - \beta\omega_z,$$
$$v' = v + \alpha\omega_z - \gamma\omega_x,$$
$$w' = w + \beta\omega_x - \alpha\omega_y,$$

the linear velocity is the same for all origins along any straight line in direction of the axis of rotation.

111. If the origin a, β, γ be such that the motion of translation of that point is in direction of the axis of rotation about it,

$$\frac{u'}{\omega_x} = \frac{v'}{\omega_y} = \frac{w'}{\omega_z},$$

$$\text{or } \frac{u + \gamma\omega_y - \beta\omega_z}{\omega_x} = \frac{v + a\omega_z - \gamma\omega_x}{\omega_y} = \frac{w + \beta\omega_x - a\omega_y}{\omega_z},$$

so that the locus of origins with this property is a straight line in direction of the axis of rotation.

112. The same result is obtained by making $u'^2 + v'^2 + w'^2$ a minimum by the alteration of a, β, γ; so that origins which give the least velocity of translation give that translation in direction of the axis of rotation.

DEF. The axis of rotation thus determined may be called the central axis of the motion by analogy to the line which receives the same name in Statics.

113. If V and Ω be the velocities of translation and rotation which express the motion of the body when the axis of the latter is in direction of the former, the velocity of any particle at distance r from this central axis is $\sqrt{V^2 + \Omega^2 r^2}$. Particles therefore which have the same velocity lie in cylindrical surfaces about the central axis.

114. The motion of a free body is thus exhibited by the velocity of any assumed point really or hypothetically in rigid union with the body, and a velocity of rotation about an axis through that point. For certain positions of the point the axis is also the direction of the velocity of rotation. Hence, in the latter view of it, the motion of the body is reduced to a screw like motion, a motion of rotation about an axis co-existing with a motion in direction of that axis.

115. A body possesses given motions of translation and rotation: to find the condition that its motion may be exhibited by rotation about a single axis.

Let the body have velocities of translation u, v, w, in directions of the rectangular axes of co-ordinates, and rotations ω_x, ω_y, ω_z about axes parallel to these originating in the point α, β, γ, whose velocity, supposing it in rigid union with the body, is the velocity of translation.

If possible let the body's motion be exhibited by a single angular velocity compounded of three rotations Ω_x, Ω_y, Ω_z about axes parallel to the co-ordinate axes and meeting in a point α', β', γ'.

The motion of the body will not be affected by pairs of opposite angular velocities equal to these being impressed about axes meeting in α, β, γ. Its motion then consists of

(1) rotations Ω_x, Ω_y, Ω_z, about the point α, β, γ,

(2) translations $\Omega_z(\beta-\beta') - \Omega_y(\gamma-\gamma')$ in direction of x,

$$\Omega_x(\gamma-\gamma') - \Omega_z(\alpha-\alpha') \ldots\ldots\ldots\ldots y,$$

$$\Omega_y(\alpha-\alpha') - \Omega_x(\beta-\beta') \ldots\ldots\ldots\ldots z.$$

If these then are equivalent to the given motions of the body,

$$\Omega_x = \omega_x, \quad \Omega_y = \omega_y, \quad \Omega_z = \omega_z,$$

$$\Omega_z(\beta-\beta') - \Omega_y(\gamma-\gamma') = u,$$

$$\Omega_x(\gamma-\gamma') - \Omega_z(\alpha-\alpha') = v,$$

$$\Omega_y(\alpha-\alpha') - \Omega_x(\beta-\beta') = w;$$

$$\therefore\ 0 = u \cdot \omega_x + v \cdot \omega_y + w \cdot \omega_z.$$

This condition of the motion being reducible to a single rotation is necessary but not sufficient: for it may be satisfied by the evanescence of the velocities of rotation while the translations remain.

116. The analogy of several of the preceding results to propositions in the theory of the equilibrium of a rigid body will hardly have escaped notice. The reader will probably have been reminded of the composition of two parallel forces (105), of the illusory case whereby a couple is presented (106), of the laws of composition of statical couples (108), of the reduction of any system of forces to a single force and a single couple (103), or to two forces (107) in all cases, and to a single force in certain cases (115), the single force being independent of the origin adopted (109) while the couple varies in general as the origin is changed and has its least value when the origin lies in the line which Poinsot designates the central axis (112).

Equations of Motion.

117. With these geometrical conceptions of the motion of a free body, we may proceed to the calculation of that motion under given forces, (1) finite, (2) impulsive.

118. The equations which determine the translation of a rigid body and its rotation about its centre of gravity are each the same as if the other did not exist.

I. *Finite forces.*

Let X, Y, Z be the finite accelerating forces parallel to the rectangular co-ordinate axes which are impressed forces on a particle of the body whose mass is m, and co-ordinates x, y, z. The effective moving forces on this particle in the same directions are $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$, $m\frac{d^2z}{dt^2}$; and since the conditions of statical equilibrium are satisfied between those forces reversed and the impressed moving forces taken throughout the whole body,

$$\therefore\ 0 = \Sigma m\left(X - \frac{d^2x}{dt^2}\right),$$

$$0 = \Sigma m\left(Y - \frac{d^2y}{dt^2}\right),$$

$$0 = \Sigma m\left(Z - \frac{d^2z}{dt^2}\right),$$

$$0 = \Sigma m\left\{\left(Z - \frac{d^2z}{dt^2}\right)y - \left(Y - \frac{d^2y}{dt^2}\right)z\right\},$$

$$0 = \Sigma m\left\{\left(X - \frac{d^2x}{dt^2}\right)z - \left(Z - \frac{d^2z}{dt^2}\right)x\right\},$$

$$0 = \Sigma m\left\{\left(Y - \frac{d^2y}{dt^2}\right)x - \left(X - \frac{d^2x}{dt^2}\right)y\right\}.$$

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the centre of gravity of the body, x', y', z' those of m referred to axes originating at the centre of gravity and parallel to the former co-ordinate axes;

$$\therefore\ x = \bar{x} + x',\quad y = \bar{y} + y',\quad z = \bar{z} + z',$$

$$\Sigma(mx') = 0,\quad \Sigma(my') = 0,\quad \Sigma(mz') = 0,$$

$$\Sigma\left(m\frac{d^2x'}{dt^2}\right) = 0,\quad \Sigma\left(m\frac{d^2y'}{dt^2}\right) = 0,\quad \Sigma\left(m\frac{d^2z'}{dt^2}\right) = 0.$$

Hence the former three equations give

$$0 = \Sigma m\left(X - \frac{d^2\bar{x}}{dt^2}\right) = \Sigma(mX) - \frac{d^2\bar{x}}{dt^2}\cdot\Sigma(m),$$

$$0 = \Sigma(mY) - \frac{d^2\bar{y}}{dt^2}\cdot\Sigma(m), \qquad (A).$$

$$0 = \Sigma(mZ) - \frac{d^2\bar{z}}{dt^2}\cdot\Sigma(m).$$

These are the equations of motion of a free molecule in the position of the centre of gravity acted on by the whole of the forces impressed on the body. Hence the equations for determining the motion of the centre of gravity are the same as if the body had no rotation.

$$\text{Again, } \Sigma\left(my\frac{d^2x}{dt^2}\right) = \bar{y}\frac{d^2\bar{x}}{dt^2}\cdot\Sigma(m) + \Sigma\left(my'\frac{d^2x'}{dt^2}\right),$$

$$\Sigma\left(mx\frac{d^2y}{dt^2}\right) = \bar{x}\frac{d^2\bar{y}}{dt^2}\cdot\Sigma(m) + \Sigma\left(mx'\frac{d^2y'}{dt^2}\right).$$

Hence the latter three equations give

$$0 = \Sigma m\left\{\left(Z - \frac{d^2z'}{dt^2}\right)y' - \left(Y - \frac{d^2y'}{dt^2}\right)z'\right\}$$

$$+ \Sigma m\left\{\left(Z - \frac{d^2\bar{z}}{dt^2}\right)\bar{y} - \left(Y - \frac{d^2\bar{y}}{dt^2}\right)\bar{z}\right\}$$

$$= \Sigma m\left\{\left(Z - \frac{d^2z'}{dt^2}\right)y' - \left(Y - \frac{d^2y'}{dt^2}\right)z'\right\} \text{ by } (A).$$

$$\text{So } 0 = \Sigma m\left\{\left(X - \frac{d^2x'}{dt^2}\right)z' - \left(Z - \frac{d^2z'}{dt^2}\right)x'\right\}, \quad (B).$$

$$0 = \Sigma m\left\{\left(Y - \frac{d^2y'}{dt^2}\right)x' - \left(X - \frac{d^2x'}{dt^2}\right)y'\right\}.$$

Now these are the equations of motion which, if the centre of gravity were a fixed point, would arise for determining the motion of the body about it (78): therefore the equations for determining the body's rotation about its centre of gravity are the same as if there were no translation.

119. Obs. When the motions of translation and rotation about the centre of gravity are said to be independent, the meaning is only that the equations belonging to these two parts of the motion are independent at any particular instant,

and not that each motion continues for a finite time as if the other had no existence. The forces which produce the rotation for instance, depend generally on the position in space of the particle on which they act, and thus they will vary with the position of the centre of gravity, so that the rotation is thus indirectly influenced by the translation. At any instant however the equations of a free particle's motion apply to the centre of gravity, and the equations of motion about a fixed point are true for the motion about the centre of gravity. (Poisson, IV. 5.)

II. *Impulsive forces.*

120. Let X, Y, Z be impulsive accelerating forces parallel to the rectangular co-ordinate axes which are impressed on a particle of the body x, y, z whose mass is m. Let u, v, w the component velocities of this particle in the same directions be suddenly changed to u', v', w' so that $m(u'-u)$, $m(v'-v)$, $m(w'-w)$ are the effective impulsive forces on m. Since the conditions of statical equilibrium are satisfied by the impressed forces and the reversed effective forces, both taken throughout the body,

$$0 = \Sigma m(X - u' + u),$$

$$0 = \Sigma m(Y - v' + v),$$

$$0 = \Sigma m(Z - w' + w),$$

$$0 = \Sigma m\{(Z - w' + w)y - (Y - v' + v)z\},$$

$$0 = \Sigma m\{(X - u' + u)z - (Z - w' + w)x\},$$

$$0 = \Sigma m\{(Y - v' + v)x - (X - u' + u)y\}.$$

Let $\bar{u}$, $\bar{v}$, $\bar{w}$ be the component velocities of the centre of gravity which by the impressed impulses are suddenly changed to $\bar{u}'$, $\bar{v}'$, $\bar{w}'$.

$$\therefore\ \Sigma m(u'-u) = \Sigma(m).(\bar{u}'-\bar{u}).$$

$$\Sigma m(v'-v) = \Sigma(m).(\bar{v}'-\bar{v}),$$

$$\Sigma m(w'-w) = \Sigma(m).(\bar{w}'-\bar{w}).$$

Hence the first three equations give

$$\left.\begin{aligned} 0 &= \Sigma(mX) - \Sigma(m).(\bar{u}' - \bar{u}) \\ 0 &= \Sigma(mY) - \Sigma(m).(\bar{v}' - \bar{v}) \\ 0 &= \Sigma(mZ) - \Sigma(m).(\bar{w}' - \bar{w}) \end{aligned}\right\} \qquad (A).$$

These are the equations of motion of a free molecule in the position of the centre of gravity acted on by the whole of the impulses impressed on the body. Therefore the equations for determining the change in the motion of the centre of gravity are the same as if the body had no rotation.

Also let $\bar{x}, \bar{y}, \bar{z}$ be co-ordinates of the centre of gravity,

$$x = \bar{x} + x', \quad y = \bar{y} + y', \quad z = \bar{z} + z'.$$

$\therefore$ the fourth equation gives

$$\begin{aligned} 0 &= \Sigma\{m(Z - w' + w)y' - (Y - v' + v)z'\} \\ &\quad + \bar{y}\Sigma m(Z - w' + w) - \bar{z}\Sigma m(Y - v' + v). \end{aligned}$$

$$\left.\begin{aligned} &= \Sigma m\{(Z - w' + w)y' - (Y - v' + v)z'\} \text{ by } (A) \\ \text{Also } 0 &= \Sigma m\{(X - u' + u)z' - (Z - w' + w)y'\} \\ 0 &= \Sigma m\{(Y - v' + v)x' - (X - u' + u)y'\} \end{aligned}\right\} \quad (B).$$

These are the equations of motion which if the centre of gravity were a fixed point would arise for determining the change of motion of the body about it (99); therefore the equations for determining the change in the body's rotatory motion about its centre of gravity are the same as if there were no translation.

121. The preceding theorems shew the advantage of adopting the centre of gravity for the point whose motion is the motion of translation (104).

122. If a rigid body move from rest by the action of a couple it will turn about its centre of gravity.

123. The proposition given in (118) supplies the equations which, when integrated and corrected by given circumstances of the motion at a given epoch, determine the translation and rotation of a rigid body under given finite forces. The equations of (120) lead to a determination of the change in the body's state under given impulses in the following manner.

124. A free body is struck by given impulses; to determine the change in its motion.

Let the motion of the body before the impulses be exhibited by translations u, v, w in direction of three co-ordinate axes through the centre of gravity with rotations ω_x, ω_y, ω_z about these same axes, and let accents denote the values into which these elements of the motion are instantaneously converted. Let $\Sigma(X)$, $\Sigma(Y)$, $\Sigma(Z)$ be the algebraic sums of the impulses resolved in directions of the axes, L, M, N their moments about the axes.

(1) The centre of gravity has its motion altered as if the impulses were applied to the whole mass collected at that point: (120)

$$\therefore\ \Sigma(m).(u'-u) = \Sigma(X),$$

$$\Sigma(m).(v'-v) = \Sigma(Y),$$

$$\Sigma(m).(w'-w) = \Sigma(Z).$$

(2) The rotation about the centre of gravity is affected as if the centre of gravity were fixed. Hence with the notation of (99),

$$L = A(\omega_x'-\omega_x) - C'(\omega_y'-\omega_y) - B'(\omega_z'-\omega_z),$$

$$M = B(\omega_y'-\omega_y) - C'(\omega_x'-\omega_x) - A'(\omega_z'-\omega_z),$$

$$N = C(\omega_z'-\omega_z) - B'(\omega_x'-\omega_x) - A'(\omega_y'-\omega_y).$$

The new elements of the motion being thus obtained, the

velocities of any particle x, y, z in directions of the co-ordinate axes are known, being

$$u' + z\omega_y' - y\omega_z' \text{ in direction of } x,$$
$$v' + x\omega_z' - z\omega_x' \ldots\ldots\ldots\ldots\ y,$$
$$w' + y\omega_x' - x\omega_y' \ldots\ldots\ldots\ldots\ z.$$

125. If two linear relations in x, y, z make these three velocities vanish, such relations are the equations to a line of particles which are at rest immediately upon the application of the impulses. Such a line is called an axis of spontaneous rotation and the initial motion is exhibited by a single rotation about this. If such a line exist the preceding expression shew that

$$0 = u'\omega_x' + v'\omega_y' + w'\omega_z'$$

agreeably with (115).

126. If the axes of co-ordinates be the principal axes at the centre of gravity, the equations of rotatory motion of a body previously at rest become

$$A\omega_x' = L, \quad B\omega_y' = M, \quad C\omega_z' = N,$$

and the condition of the motion being one of rotation about a single axis determinate in position, gives

$$0 = \frac{L.\Sigma(X)}{A} + \frac{M.\Sigma(Y)}{B} + \frac{N.\Sigma(Z)}{C}.$$

127. Since the direction cosines of the instantaneous axis are proportional to ω_x', ω_y', ω_z' and therefore to $\frac{L}{A}, \frac{M}{B}, \frac{N}{C}$, while the direction cosines of the resultant impressed impulse are proportional to $\Sigma(X)$, $\Sigma(Y)$, $\Sigma(Z)$, the preceding equation states that the spontaneous axis, if it exists, is perpendicular to the resultant impulse when the impulses admit of reduction to a single blow. (Cambridge Math. Journal, Vol. IV.)

128. If the axis of spontaneous rotation were a fixed axis, the point where the impulse is applied is a centre of percussion relatively to it (60).

SECTION VI.

MOTION OF A SYSTEM OF RIGID BODIES.

129. When a system of rigid bodies in motion influence one another by certain given connections, by attraction, or contact, or the action of strings or rods for instance, they will exert two and two opposite forces on one another. Each body when the forces thus in action upon it are taken into account, may be treated as a free body, and the unknown mutual forces are to be eliminated between the systems of mechanical equations which the several bodies supply and the geometrical equations which express the relations of the parts of the system to one another or to other bodies.

Similarly if constraints extraneous to the system act upon parts of it, if the forces which these constraints exert be represented by certain assumed quantities and thus taken into account, the system may then be treated as dynamically free, though subject still to the geometrical conditions of position which the said restraints impose.

Thus the equations of motion of each body of the system may be constructed according to the principles of the preceding sections. The equations which arise from mechanical principles are limited in number, six namely at most for each body of the system, because six equations are the necessary and sufficient conditions of equilibrium of a rigid body. The equations which geometry supplies are not limited in number, and, if the problem be determinate, will with the dynamical equations already mentioned make up a number of equations equal to the number of quantities whose determination is requisite for the complete solution of the problem.

130. Since each body of the system would be separately at equilibrium if the effective force on each molecule were applied in a reversed direction (30), the whole system would in this case be at equilibrium, and its state of equilibrium would not be disturbed by supposing any or all of its parts rigidly united. Hence the equations of equilibrium of the whole system may be constructed as if it were a single rigid body, mutual impressed forces being hereby left out of consideration.

131. If two bodies press against one another, whereof one at least is smooth, and if a finite area of the surface of each is in contact, the resultant mutual action will be at some unknown point of this common area, the determination of which is a part of the problem. If their surfaces have only a point in common, the resultant action is in the normal to either of the surfaces at this point, supposing that there is a determinate normal to one of the surfaces at this point or to both of them in common. A cube pressing against a sphere first at a corner, secondly at a point of one of its faces, is an instance of these kinds of mutual action respectively. The geometrical condition of this case is that the velocity of the two points of contact in direction of the mutual action is the same. If the normal to neither surface at the point of contact is determinate, as when two cubes press against one another at an angular point of each, the mutual action is unknown in direction as well as magnitude, and part of the complete solution of the problem will be the determination of its direction as well as its magnitude.

132. When the surfaces of the bodies which press against one another are both rough, in addition to the action already described there will be mutual force perpendicular to it caused by friction. When the points in contact have different velocities the bodies are said to slide, and in this case the friction $= \mu \times$ normal pressure, where μ is a constant dependent on the nature of the two substances only, and independent of their velocities, and of the shape or extent of the surfaces in

contact*. When the surfaces touch at a point and when the point of contact has the same velocity in each, the bodies are said to roll upon one another. In this case the amount of friction is unknown, and will be one of the circumstances which the solution of the problem is to determine.

The preceding statement includes the case of a body pressing on a fixed surface. The geometrical condition of sliding will be that the point or points of contact in the body have no velocity perpendicular to the fixed surface at the point or respective points of contact, and the condition of rolling that the point or points of contact of the rolling body have no velocity.

133. The forces arising from mutual action which have been hitherto considered are called finite forces, having this property, that they can produce no finite effect unless they have acted during a finite time. Each particle of a system under the action of forces of this kind in passing from one velocity to another has had in succession all degrees of velocity between these, and its path has been a continuous curve. Now cases of mutual action occur to which this description will not apply. If two bodies strike one another, if they be connected by a string which after being loose becomes at last tight, if an explosion occur in the interior of a body, then at the instant of these events, the motion of any particle generally undergoes a sudden change both as to velocity and direction, a change not indeed really instantaneous, but occupying a time which is inappreciable. The forces which act in this case are impulsive forces, their definition being that they can produce a finite effect in an inappreciable time.

If the laws of the internal constitution of bodies and of the action of their molecules upon one another were known, such circumstances as have been mentioned would introduce no new feature. At any time during a collision or explosion the force acting on any molecule would be known and its effect might

* These laws have been verified with great accuracy by M. Morin. An abstract of his results is given by Mr Moseley. *Illustrations of Mechanics.* APPENDIX.

be computed. But in want of such knowledge we are obliged to reason upon the entire impulsive action from the beginning to the end of its existence. This duration we neglect as inappreciable, and measure an impulsive force by the momentum which it is capable of generating instantaneously on a free particle on which it acts.

134. In problems where mutual impulsive action arises the elasticity of the bodies is an important element. The collision of two bodies produces a compression of each to an extent depending on their hardness. Inelastic bodies are those which after being compressed or extended have no power of regaining their shapes: with such bodies therefore the condition of the end of impact is that the points in contact have the same velocity in the direction in which the impulsive action has taken place. Elastic bodies again exert a further action on one another additional to this, and bearing a ratio to it which depends only on their materials. Thus if R be the impulse between two bodies, R_1 the impulse which would have existed if they had been inelastic, $R = (1 + e) R_1$, where e depends only on the substances of the two bodies.

135. In determining the instantaneous change of the motion of bodies produced by impulse, finite forces may be left out of consideration, for the effect of these latter is not developed except in a finite time. The examination of the motion of bodies in the course of which mutual impulsive action occurs will usually consist of three stages, a calculation (1) of the motion before the impulses under the finite forces which may be in action, whereby the state of motion of the system immediately before the impulses is known, (2) of the state of motion into which the former is suddenly altered by the impulses, and (3) of the subsequent motion under the finite forces which may still continue in action.

136. Cases where the mutual force approaches to the nature of impulse as here defined, but cannot be safely regarded as instantaneous, as for instance when a ram drives a

pile, are discussed in Professor Whewell's Mechanics of Engineering. Chap. XII.

137. In examining the action of impulsive forces we are led to consider bodies which do not exactly fulfil the condition of rigidity, but are capable of sustaining small compressions and again of assuming their former forms by their elasticity. The principle of D'Alembert, it will be remembered, includes cases like these. (35).

138. The general method of determining the motion of a system of bodies under the action of one another, or of given fixed restraints, such as surfaces along which they roll or slide has been already described (129). The differential equations will be of the second order, and the unknown forces of constraint or mutual action must be removed by elimination before the integration can be attempted. This elimination may often be avoided by aid of the general principles contained in the following articles. These principles, which can be enunciated as practical rules, enable us to arrive at once at the result of one integration of the differential equations of motion, and generally reduce the problem to the solution of a differential equation of the first order.

It must be observed that these principles give no result which cannot be drawn from the differential equations of motion. They do no more than expedite the analytical operation of solution. This remark shews how any uncertainty in the application of these principles can always be resolved, and any results obtained from the use of them be checked.

I. *Conservation of the motion of the centre of gravity.*

139. The centre of gravity of a system of moving bodies connected in any manner has the same motion as if the mass of the whole were collected at that point and acted upon by the impressed moving forces which are extraneous to the system.

If the effective force on every particle act upon it in a direction contrary to its real direction, the conditions of equi-

librium of each body are satisfied between these and the whole of the impressed forces (30). The conditions of equilibrium are therefore also satisfied if the bodies become united into one rigid body, in which case the mutual forces become molecular and so cease to appear in those equations. In this case if P, Q, R be the sums of the extraneous impressed forces resolved parallel to the rectangular co-ordinate axes of x, y, z, m the mass of a particle x, y, z, M the mass of the system, $\bar{x}$, $\bar{y}$, $\bar{z}$ its centre of gravity,

$$0 = P - \Sigma\left(m\frac{d^2x}{dt^2}\right) = P - M.\frac{d^2\bar{x}}{dt^2},$$

$$0 = Q - \Sigma\left(m\frac{d^2y}{dt^2}\right) = Q - M.\frac{d^2\bar{y}}{dt^2},$$

$$0 = R - \Sigma\left(m\frac{d^2z}{dt^2}\right) = Q - M.\frac{d^2\bar{z}}{dt^2}.$$

These are the equations of motion of a particle whose mass is M, at $\bar{x}$, $\bar{y}$, z, under the forces P, Q, R. Hence the motion of the centre of gravity of the system is that of a particle in its position under the action of the whole of the impressed forces which are extraneous to the system.

140. Since this investigation is independent of the nature and magnitude of internal forces, the result is true if any collisions among the parts or explosions arise. The motion of the centre of gravity will not be influenced by action of this kind.

141. If no forces extraneous to the system are acting, the centre of gravity is either at rest or moves uniformly in a straight line. This invariability of the motion of the centre of gravity has given rise to the name of the principle now under consideration.

II. *Conservation of areas.*

142. If a system of bodies be in motion under forces which, if the bodies were rigidly connected, would not tend to produce rotation about a certain fixed straight line, then the sum of the products of the mass of each particle and the area which its projection on any plane perpendicular to that line describes about the line is proportional to the time during which the motion is considered.

Let the fixed straight line be the axis of x, m the mass of a particle whose co-ordinates at time t are x, y, z; X, Y, Z the accelerating forces impressed upon it in directions of the co-ordinates. Since the forces would not tend to produce rotation in the system about the axis of x, if the parts of it were rigidly connected,

$$\Sigma m(Zy - Yz) = 0.$$

$$\therefore \Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = 0, \quad (130).$$

$$\therefore \Sigma m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) = c, \text{ a constant.}$$

Let $\frac{1}{2}A_x$ be the area which the projection of m on the plane yz describes about x from the fixed time t_1 to the time t,

$$\therefore \Sigma\left(m\frac{dA_x}{dt}\right) = c,$$

$$\Sigma(mA_x) = c(t - t_1);$$

or if T be the duration between the instants represented by t and t_1,

$$\Sigma(mA_x) = cT \propto T.$$

143. The result of this proposition continues to apply so long as the condition $\Sigma m(Zy - Yz) = 0$ is preserved, and therefore is not affected by any changes which the system may

undergo from any mutual internal forces, whatever be their intensity or duration, such as those which arise in collision between parts of the system or explosion.

144. If the three conditions

$$\Sigma m(Zy - Yz) = 0,\ \Sigma m(Xz - Zx) = 0,\ \Sigma m(Yx - Xy) = 0, \quad (1)$$

are simultaneously satisfied, so that if the system were rigidly united the forces would not tend to produce motion about the origin, then the sum of the masses of every particle each multiplied into the area which its projection on any plane through the origin describes about the origin, is proportional to the time during which the motion is considered.

For if A_x, A_y, A_z designate the doubles of the areas which the particle m describes in the interval T on the co-ordinate planes, to each of which the reasoning of (142) now applies,

$$\frac{\Sigma(mA_x)}{c_1} = \frac{\Sigma(mA_y)}{c_2} = \frac{\Sigma(mA_z)}{c_3} = T;$$

and if A be the area similarly described in projection on a plane whose direction cosines are l, m, n,

$$\begin{aligned}\Sigma(mA) &= \Sigma m(A_x . l + A_y . m + A_z . n)\\ &= (c_1 l + c_2 m + c_3 n)\, T \propto T.\end{aligned}$$

145. The expression $c_1 l + c_2 m + c_3 n$ is a maximum while $l^2 + m^2 + n^2 = 1$, if

$$\frac{l}{c_1} = \frac{m}{c_2} = \frac{n}{c_3} = \frac{1}{\sqrt{c_1^2 + c_2^2 + c_3^2}}.$$

Since these values of the direction cosines are invariable, the plane on which the sum of the masses of the molecules each multiplied into the projected area which it describes thereon in a given time, is the greatest possible, remains permanent in position so long as the original conditions (1) are fulfilled. The plane is hence called the invariable plane of the system with reference to the assumed origin of co-ordinates.

In examining the system of the universe where all the actions are mutual and the conditions (1) therefore obtain, this plane supplies a plane of reference which is unaffected by any alterations which take place in the constitution of any parts of the system. Its utility in the particular case of a single body in place of a system of bodies has been seen in (86, 87).

146. $\Sigma(mA) = 0$ if $c_1 l + c_2 m + c_3 n = 0$, that is, if the projections are made on a plane perpendicular to the invariable plane.

147. Since the functions c_1, c_2, c_3 are generally different when the system is referred to different origins, with respect to which the moments of the forces vanish, hence the invariable plane generally differs in position when it is determined in reference to these different origins.

If c_1', c_2', c_3' be the values of the parameters when the origin is moved to the point a, b, c; M the mass of the whole system and $\bar{u}$, $\bar{v}$, $\bar{w}$ the velocities of its centre of gravity parallel to the co-ordinate axes, which are supposed to retain permanent directions,

$$c_1' = \Sigma m\left(\overline{x - c} \cdot \frac{dy}{dt} - \overline{y - b}\,\frac{dx}{dt}\right),$$

$$= c_1 - M(c\bar{v} - b\bar{w}),$$

$$c_2' = c_2 - M(a\bar{w} - c\bar{u}),$$

$$c_3' = c_3 - M(b\bar{u} - a\bar{v}).$$

Hence the position of the invariable plane is the same for all origins along a line in direction of the motion of the centre of gravity. If the centre of gravity has no velocity, the position of the invariable plane is independent of the origin selected.

148. The invariable plane at the origin a, b, c, is

$$X\{c_1 - M(c\bar{v} - b\bar{w})\}$$
$$+ Y\{c_2 - M(a\bar{w} - c\bar{u})\}$$
$$+ Z\{c_3 - M(b\bar{u} - a\bar{v})\} = ac_1 + bc_2 + cc_3.$$

149. The sum of the masses of each particle of the system multiplied each into the area which its projection describes in a given time on a plane through the origin a, b, c, is greatest, as has been seen, when that plane is the invariable plane, and in that case $= \sqrt{c_1'^2 + c_2'^2 + c_3'^2}$. If now the origin a, b, c be such that this expression, the maximum with respect to different planes at that origin, may be the minimum with respect to different origins,

$$0 = c_1' dc_1' + c_2' dc_2' + c_3' dc_3',$$

$$0 = c_1' (\bar{v} dc - \bar{w} db),$$

$$+ c_2' (\bar{w} da - \bar{u} dc),$$

$$+ c_3' (\bar{u} db - \bar{v} da),$$

whence since da, db, dc are independent,

$$\frac{c_1'}{\bar{u}} = \frac{c_2'}{\bar{v}} = \frac{c_3'}{\bar{w}},$$

or the origin required has this property, that the invariable plane belonging to it is perpendicular to the direction of the motion of the centre of gravity. Such an origin will have for its locus a line in direction of the motion of the centre of gravity.

150. If $\bar{x}$, $\bar{y}$, $\bar{z}$ be co-ordinates of the centre of gravity of the system, x', y', z' those of m referred to parallel axes originating at the centre of gravity,

$$\Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \left(\bar{x} \frac{d\bar{y}}{dt} - \bar{y} \frac{d\bar{x}}{dt} \right) \Sigma (m)$$

$$+ \Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right).$$

Hence the function $\Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right)$ is evaluated by adding together its values obtained on the separate suppositions, (1) that the whole system was condensed at its centre of gravity

into a point, (2) that the system is in motion about its centre of gravity fixed, the two portions of the function in short due to the motions of translation and rotation.

III. *Principle of Vis Viva.*

151. DEF. The vis viva of a particle in motion is the product of its mass and the square of its velocity.

The vis viva of a system in motion is the sum of the vires vivæ of its particles.

152. If X, Y, Z be the accelerating forces on a free particle in directions parallel to the co-ordinate axes, m the mass of the particle, the change of its vis viva in passing from one position to another in the course of its motion

$$= 2m\int(Xdx + Ydy + Zdz),$$

the limits of integration being those two positions.

153. If a rigid body be revolving about an axis with an angular velocity ω, and if r be the distance from that axis of a molecule whose mass is m, the vis viva of the particle is $m\omega^2r^2$ (39), and the vis viva of the body is $\Sigma(mr^2).\omega^2$, or the body's moment of inertia about the axis multiplied by the square of the angular velocity.

154. The vis viva of a body is the sum of the vires vivæ which it would separately have if it possessed the motion of translation of the centre of gravity or the motion of rotation about that centre separately.

Let x, y, z be the co-ordinates at time t of a particle of the body of mass m, $\bar{x}$, $\bar{y}$, $\bar{z}$ those of the centre of gravity, x', y', z' the co-ordinates of m referred to axes through the centre of gravity parallel to the former;

$$\therefore\ x = \bar{x} + x', \quad y = \bar{y} + y', \quad z = \bar{z} + z',$$

$$\Sigma(mx') = 0, \quad \Sigma(my') = 0, \quad \Sigma(mz') = 0.$$

$$\Sigma\left(m\frac{dx'}{dt}\right)=0, \quad \Sigma\left(m\frac{dy'}{dt}\right)=0, \quad \Sigma\left(m\frac{dz'}{dt}\right)=0.$$

$$\therefore \Sigma\left\{m\left(\frac{dx}{dt}\right)^2\right\}=\left(\frac{d\bar{x}}{dt}\right)^2.\Sigma(m)+\Sigma\left\{m\left(\frac{dx'}{dt}\right)^2\right\}.$$

$\therefore$ the vis viva of the body

$$=\Sigma m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}$$

$$=\left\{\left(\frac{d\bar{x}}{dt}\right)^2+\left(\frac{d\bar{y}}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}\Sigma(m)$$

$$+\Sigma m\left\{\left(\frac{dx'}{dt}\right)^2+\left(\frac{dy'}{dt}\right)^2+\left(\frac{dz'}{dt}\right)^2\right\}.$$

Of the two terms of this expression the former is the vis viva which the body would have if every particle possessed the velocity of the centre of gravity, the latter is the vis viva which the body would have if it were moving about the centre of gravity fixed. The sum of these is the vis viva of the body.

155. Statement and proof of the principle of vis viva.

If a system of rigid bodies be in motion under the action of finite forces, the geometrical connexion of the different parts being expressed by equations which do not contain the time explicitly and remain invariable in form, the change of vis viva of the system in passing from any one position to another is the same as if the particles had been free and had been acted upon by the same impressed forces through the same spaces.

Let x, y, z be rectangular co-ordinates of a particle of the system whose mass is m, X, Y, Z the accelerating impressed forces on the same in directions of the co-ordinate axes.

Then $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$, $m\frac{d^2z}{dt^2}$ are the effective moving forces on this particle in the same directions, and if such effective forces be applied in reversed directions to the particles to which they belong, the conditions of equilibrium are satisfied among the reversed forces and the impressed forces (30, 130).

Let the system receive a change of position in consistence with the connexions of its parts and let the displacement of m parallel to the co-ordinate axes be ultimately expressed by δx, δy, δz as the alteration of position is reduced indefinitely. Then by the principle of virtual velocities

$$0 = \Sigma m\left\{\left(X - \frac{d^2x}{dt^2}\right)\delta x + \left(Y - \frac{d^2y}{dt^2}\right)\delta y + \left(Z - \frac{d^2z}{dt^2}\right)\delta z\right\}.$$

Let $L = 0$ be one of the equations expressing the geometrical connexions of the system by exhibiting a relation among the co-ordinates of certain of its particles. Since the displacement already imagined is consistent with such connexions,

$$0 = \frac{dL}{dx}.\delta x + \frac{dL}{dy}.\delta y + \ldots\ldots\ldots \quad (1).$$

Now in the actual motion of the system such equations as $L = 0$ continue true by the limitation of the statement of the motion considered. Hence L must be invariable under a change of time and consequent changes of position;

$$\text{or } 0 = \frac{dL}{dt}.\delta t + \frac{dL}{dx}.\frac{dx}{dt}.\delta t + \frac{dL}{dy}.\frac{dy}{dt}.\delta t + \ldots\ldots\ldots \quad (2).$$

Hence if $\frac{dL}{dt} = 0$, and if the same be the case in the other equations of its class, we can in consistence with equations of the kind (1) and (2) assume

$$\delta x = \frac{dx}{dt}\delta t, \qquad \delta y = \frac{dy}{dt}.\delta t\ldots\ldots\ldots,$$

or by reason of the expressions of connexion between the

parts not involving the time explicitly, we may ultimately identify the hypothetical displacements which the principle of virtual velocities allows with the real changes of position which the actual motion produces. Denoting these latter by dx, dy, ... we then have

$$\Sigma m\left(\frac{d^2x}{dt^2}\cdot\frac{dx}{dt}+\frac{d^2y}{dt^2}\cdot\frac{dy}{dt}+\frac{d^2z}{dt^2}\cdot\frac{dz}{dt}\right)\delta t=\Sigma m(Xdx+Ydy+Zdz);$$

$$\therefore\ \Sigma m\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}$$

$$=2\Sigma m\int(Xdx+Ydy+Zdz),\quad (A),$$

the integral being taken between the two positions between which the system is considered to pass, so that the correction involved is the vis viva in the former of those positions. The former member of the equation is the vis viva of the system in the latter of the two positions.

Now $2m\int(Xdx+Ydy+Zdz)$ is the change which would have arisen in the vis viva of the particle m if it had been free and had passed from one to the other of its positions under the accelerating forces X, Y, Z, (152). Hence the change of the vis viva of the system between the two positions is the same as if the particles had been free and had been acted on by the same impressed forces through the same spaces.

156. A reference to the principle of virtual velocities on which the preceding proposition has been demonstrated, will shew that forces of the following kinds will not appear in the latter member of equation (A).

(1) Mutual forces arising from the pressure of parts in contact or the tensions of inextensible strings.

(2) Reactions of fixed rough surfaces on which parts of the system roll, or of fixed axes, because the points where these forces act have no velocity (132).

(3) Reactions of fixed smooth surfaces along which parts of the system slide, because the points where these forces act have no velocity except in directions perpendicular to the directions of the forces.

By evading forces of these kinds the principle of vis viva shews its usefulness, and expedites the solution of problems. It must be remembered that the application of the principle is limited to cases where the actual motion is of a kind which may be employed as a hypothetical change of position admissible in using the principle of virtual velocities.

157. The expression $Xdx + Ydy + Zdz$ can be integrated as a perfect differential whenever the forces tend to fixed centres at finite distances, and are functions of the distances from the centres, under which characters all forces existing in nature except molecular force will be comprehended.

For if X, Y, Z be components of a force tending to a centre whose co-ordinates are a, b, c and acting at x, y, z, and in intensity a function of its distance r from this centre,

$$\frac{X}{x-a} = \frac{Y}{y-b} = \frac{Z}{z-c} = -\frac{\phi(r)}{r},$$

while $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$.

$$\therefore (x-a)dx + (y-b)dy + (z-c)dz = r\,.\,dr,$$

$$\therefore Xdx + Ydy + Zdz = -\phi(r)\,.\,dr.$$

If the forces X, Y, Z are the components of several forces of this character, $Xdx + Ydy + Zdz$ is the sum of a corresponding number of perfect derivatives.

158. Also $\Sigma m(Xdx + Ydy + Zdz)$ will be a perfect differential as far as it results from the attractions of one particle of the system on another at a finite distance.

For if x, y, z be co-ordinates of m, x', y', z' of m', then the terms of the function which depend on their mutual attraction are

$$m\left(m'\frac{x'-x}{r}dx + m'\frac{y'-y}{r}dy + m'\frac{z'-z}{r}dz\right)\phi(r)$$

$$+ m'\left(m\frac{x-x'}{r}dx' + m\frac{y-y'}{r}dy' + m\frac{z-z'}{r}dz'\right)\phi(r)$$

$$= -mm'\phi(r).dr,$$

since

$$r^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2.$$

159. Hence under the forces which exist in nature, change in vis viva

$$= -2\Sigma m\int\phi(r)dr$$

$$= \Sigma mf(x, y, z) - \Sigma mf(x_0, y_0, z_0),$$

where x_0, y_0, z_0 are the co-ordinates in the initial position of the molecule m whose present co-ordinates are x, y, z, the symbol Σ expressing a geometrical integration performed through the whole system in either position.

Since the change of vis viva depends only on the initial and final positions of the system and on the law of the forces to which it is subject, it is therefore independent of the paths which the molecules have described in passing from the one position to the other.

160. When a force acts upon a body in motion the efficiency or 'work' of the force between two positions of the body is measured by the product of the force and the space through which its point of application moves in its direction, when the force and its direction are constant: when either of these varies, the efficiency is obtained by integrating the elementary value of it obtained on the preceding definition.

161. Let mX, mY, mZ be the pressures acting at time t parallel to the co-ordinate axes on a particle m of the system. Then as the system passes into a consecutive position

$$m(Xdx + Ydy + Zdz)$$

is the efficiency expended by the forces applied to m, and

$$\Sigma m(Xdx + Ydy + Zdz)$$

is the efficiency expended on the whole system.

Hence the efficiency expended between any two positions of the system $= \Sigma m \int (Xdx + Ydy + Zdz)$, the integral being taken from one of those positions to the other, and if the motion of the system obeys the principle of vis viva, this is half the change of vis viva in passing between the same states.

162. The principle of vis viva in the form of the last article has an important use in calculating the motion of machines which consist of rigid parts united together and communicating motion one to another, provided the nature of the connexions fulfils the conditions of motion by which the truth of the principle of vis viva is limited. In cases like these the function $\Sigma m \int (Xdx + Ydy + Zdz)$ consists of three parts arising (1) from the forces applied to the machine as the primary cause of its motion, (2) the internal forces from the weight and friction of the parts which are not included in the exceptions of (156), and (3) the reactions of the bodies on which the machine is designed to act and in which it produces motion. Forces of the first kind tend to augment the vis viva of the system, while those of the latter two generally tend to diminish it, and therefore the efficiency of these latter is in general negative. Hence the change of vis viva of the machine between two states is double the excess of the efficiency applied to it by the original moving agent over the sum of the efficiency consumed within itself and that exhibited on the objects on which it is employed.

When a machine is working uniformly the efficiency applied to it in any interval equals the sum of the efficiency

consumed internally and that transmitted effectively. When the action of the machine is periodic so that its vis viva returns to the same state after given intervals, the same statement applies to each of such intervals. If the machine was at first at rest, efficiency must have been expended in generating the vis viva which is thence maintained uniform or periodic in the manner above mentioned.

Since the internal resistances of a machine usually consume part of the efficiency applied to it, the efficiency of an agent is generally impaired when it is transmitted through machinery, and in the most favourable case can only equal the efficiency exhibited. A machine therefore is advantageous not in increasing the efficiency of the agent which works it, but in applying the power of the agent more conveniently.

163. Hence no machine can by itself maintain its own motion for an indefinite time and also do work, or what is understood by perpetual motion is physically impossible under the forces presented in nature. See a paper by Mr Airy on this subject in the third volume of the Transactions of the Cambridge Philosophical Society.

164. A fourth general principle of the motion of a system called the principle of least action, which has generally been added to those already given, is here omitted, because as an auxiliary in the solution of problems it is nearly useless, and as a theory, it is only a particular case of a more general principle developed by Sir W. R. Hamilton, in the Philosophical Transactions of 1834 and 1835, and to these students for whom such reading is expedient are referred.

APPENDIX.

THE following proposition, which is frequently required when two systems of rectangular co-ordinate axes are employed, is introduced in this place to avoid interrupting the subject of the section in which it arose.

The convertibility of the two groups of equations between the direction cosines in (63) and (64) is manifest from each being the expression of rectangularity of a system of axes referred to another rectangular system, and one or the other group of equations arises as one or the other system of axes is made the system of reference. The following however is an algebraic proof.

$$(a_1^2 + b_1^2 + c_1^2 - 1)^2 + (a_2^2 + b_2^2 + c_2^2 - 1)^2 + (a_3^2 + b_3^2 + c_3^2 - 1)^2$$

$$+ 2(a_3a_2 + b_3b_2 + c_3c_2)^2 + 2(a_1a_3 + b_1b_3 + c_1c_3)^2 + 2(a_1a_2 + b_1b_2 + c_1c_2)^2$$

$$= \Sigma(a^4) + 2\Sigma(a_1^2b_1^2) - 2\Sigma(a^2) + 3$$

$$+ 2\Sigma(a_2^2a_3^2) + 4\Sigma(a_2a_3\,b_2b_3)$$

$$= \Sigma(a^4) + 2\Sigma(a_2^2a_3^2) - 2\Sigma(a^2) + 3$$

$$+ 2\Sigma(a_1^2b_1^2) + 4\Sigma(a_2b_2\,a_3b_3)$$

$$= (a_1^2 + a_2^2 + a_3^2 - 1)^2 + (b_1^2 + b_2^2 + b_3^2 - 1)^2 + (c_1^2 + c_2^2 + c_3^2 - 1)^2$$

$$+ 2(b_1c_1 + b_2c_2 + b_3c_3)^2 + 2(a_1c_1 + a_2c_2 + a_3c_3)^2 + 2(a_1b_1 + a_2b_2 + a_3b_3)^2.$$

Hence if one member of this equality vanishes by virtue of the one of the two groups of equations which is given, the six squares which form the other member being in consequence separately zero, give the six equations which form the other group whose truth is to be established.

EXAMPLES.

REFERENCES are generally attached to the following examples in order to suggest the principles which they are severally intended to illustrate or apply. The order of arrangement adopted in the preceding sections of this book has been followed as far as possible.

I. *Geometrical properties of a rigid body.*

1. Evaluations of the functions $\Sigma(myz)$, $\Sigma(mxz)$, $\Sigma(mxy)$ in particular instances, the axes being rectangular and the body homogeneous; M the mass of the body.

If ρ be the density of the body at the point x, y, z, these functions are equivalent to the definite integrals $\int_x\int_y\int_z \rho yz$, $\int_x\int_y\int_z \rho xz$, $\int_x\int_y\int_z \rho xy$ respectively, the limits of integration including the whole body.

(α) A circular lamina in the plane xy, a, b co-ordinates of its centre.

$$\Sigma(myz) = 0, \quad \Sigma(mxz) = 0, \quad \Sigma(mxy) = Mab.$$

(β) A lamina in the plane xy whose boundary is symmetrical with regard to the axes of x and y.

$$\Sigma(myz) = 0, \quad \Sigma(mxz) = 0, \quad \Sigma(mxy) = 0.$$

(γ) A right-angled triangle whose sides a, b are in the axes of x and y,

$$\Sigma(myz) = 0, \quad \Sigma(mxz) = 0, \quad \Sigma(mxy) = \tfrac{1}{12}Mab.$$

(δ) A straight line where a, b, c and a', b', c' are the co-ordinates of its extremities,

$$\Sigma(myz) = \tfrac{1}{6}M\{2(bc + b'c') + bc' + b'c\}$$
$$\Sigma(mzx) = \tfrac{1}{6}M\{2(ac + a'c') + ac' + a'c\}$$
$$\Sigma(mxy) = \tfrac{1}{6}M\{2(ab + a'b') + ab' + a'b\}.$$

(ϵ) A sphere whose centre is a, b, c,

$$\frac{\Sigma(myz)}{bc} = \frac{\Sigma(mzx)}{ac} = \frac{\Sigma(mxy)}{ab} = M.$$

(ζ) In a plane lamina referred to such rectangular axes in its plane that lines parallel to them through the centre of gravity $(\bar{x}\,\bar{y})$ are lines of symmetry,

$$\Sigma(mxy) = M\bar{x}\bar{y}.$$

(η) A circular right cone is referred to its vertex as origin and such axes that the plane of xy passes through its axis and a generating line is the axis of x.

$$\Sigma(mxy) = M \sin\alpha \cdot \cos\alpha \left\{\frac{3}{5}a^2 - \frac{3}{20}r^2\right\},$$

where α is the semi-vertical angle of the cone, a its altitude and r the radius of its base.

2. In any body $\Sigma(mz^2) \cdot \Sigma(my^2) > \{\Sigma(myz)\}^2$.
$\Sigma(mx^2) \cdot \Sigma(mz^2) > \{\Sigma(mzx)\}^2$.
$\Sigma(mx^2) \cdot \Sigma(my^2) > \{\Sigma(mxy)\}^2$.

3. The moment of inertia of a circular arc about an axis through one extremity of it perpendicular to its plane

$$= 2Mr^2\left\{1 - \frac{\sin\theta}{\theta}\right\},$$

r being the radius and θ the angle subtended. (8).

4. The moment of inertia of the whole arc of a circle about a diameter $= \frac{1}{2} Mr^2$. (9).

5. An oblate spheroid consists of similar strata each indefinitely thin and uniform in density. Its moment of inertia about its axis of figure

$$= \frac{8}{3} \pi \sqrt{1 - e^2} \int_a^0 \rho \, a^4,$$

where ρ is the density of the stratum of which $2a$ is the equatoreal diameter, e the eccentricity of the strata.

Let $\rho = c \frac{\sin na}{a}$ for instance.

6. The sum of the moments of inertia of a rigid body about any three rectangular axes at the same origin is the same. (10).

7. If a point is taken in one of a system of three principal axes, axes through this point parallel to the other two original principal axes form a new system of principal axes when the first axis passes through the centre of gravity of the body, but not generally in other cases.

8. If a body satisfy the conditions $\Sigma (mx^2) = \Sigma (my^2) = \Sigma (mz^2)$ when referred to a system of principal axes, any other rectangular system of axes with the same origin are also principal axes.

9. The principal axes of a right-angled triangle at the right angle are one perpendicular to its plane, and two others in its plane and inclined to its sides at the angle

$$\frac{1}{2} \tan^{-1} \frac{ab}{2(a^2 - b^2)}. \quad (17, 18).$$

10. If a point in the circumference of an elliptic board

be made origin, the position of the principal axes in its plane is given by

$$\tan 2\theta = \frac{2\alpha\beta}{\alpha^2 - \beta^2 - \frac{1}{4}(b^2 - a^2)},$$

where α, β are co-ordinates from the centre of the assumed origin. (17).

11. The moment of inertia of a cube about a diagonal $= \frac{1}{6}Ma^2$, (25), and about a diagonal of one of its faces $= \frac{5}{12}Ma^2$, a being an edge of the cube. (25, 5).

12. The moment of inertia of a circular right cone about a generating line $= \frac{3}{20}Mc^2 . \frac{6a^2 + c^2}{a^2 + c^2}$, a being the altitude and c the radius of the base. (24).

13. The moment of inertia of an elliptic area about any diameter is proportional to the square of that diameter inversely. (21).

14. The moment of inertia of an ellipsoid about a diameter $= \frac{1}{5}(a^2 + b^2 + c^2 - p^2)M$, where p is the perpendicular from the centre on the tangent plane perpendicular to the diameter. (21).

15. The moment of inertia of a spheroid about all axes through one of its poles is the same. Its axes are in the ratio of $1 : \sqrt{6}$. (25).

16. A circular right cone whose altitude is half the radius of its base has the same moment of inertia about every axis through its vertex. (25).

II. *Motion of a body about a fixed axis.*

1. A body revolving about an axis has a velocity $a \sec^2\theta$. It will make each revolution in half the time in which it would revolve with the uniform angular velocity a. (38).

2. If a rigid body has an angular velocity ω about the straight line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, the velocities of a point x, y, z in direction of the co-ordinate axes are

$$\frac{mz - ny}{\sqrt{l^2 + m^2 + n^2}}\omega, \quad \frac{nx - lz}{\sqrt{l^2 + m^2 + n^2}}\omega, \quad \frac{ly - mx}{\sqrt{l^2 + m^2 + n^2}}\omega.$$

3. A hemispherical surface whose radius is r makes small oscillations about a diameter of its base in time $\pi\left(\frac{4r}{3g}\right)^{\frac{1}{2}}$. (49).

4. A wire is bent into the form of an arc of a circle, and makes small oscillations about a horizontal axis through its middle point perpendicular to its plane in 1·5708 seconds. The radius of the arc is four feet nearly. (49).

5. A sphere revolves about a given chord as a vertical axis. Find the pressure on the axis and explain the result when the chord becomes a tangent. (42).

6. A square revolves about a vertical edge. Shew whether under any assumption of angular velocity the resultant pressure on the axis can ever act (1) through the highest point, (2) at the middle point. (42).

7. A rod revolves in a vertical plane about its upper end. Shew in what positions of the rod the pressure on the point of support is (1) in direction of the rod, (2) perpendicular to the rod. (46).

8. A cylinder whose axis $= \sqrt{\frac{3}{2}} \times$ radius of base, is capable of revolving about a horizontal diameter of its base and falls from the position wherein the axis is horizontal. If the axis and the direction of the pressure upon the fixed diameter make with the horizon the angles ϕ and ψ respectively in any position of the body, $4\tan(\psi - \phi) = \cot\phi$. (46).

9. When the centre of gravity of a revolving body is vertically under the horizontal axis the pressure on the axis resolved horizontally does not vanish unless the origin can be so taken that $\Sigma(mxz) = 0$, z being the axis of revolution and x also horizontal. (42).

A hemisphere revolves about an axis which coincides with a diameter of its base and is inclined at an angle α to the vertical. If it swings through 180^0 the whole pressure on the axis in the lowest position

$= \frac{1}{64}\{(109 \sin \alpha)^2 + (64 \cos \alpha)^2\}^{\frac{1}{2}} \times$ weight of the body. (42).

10. An oblate spheroid revolves about a horizontal axis. Find the position of the axis when the time of small oscillation is the least possible. (49).

11. A cube revolves about a horizontal edge so as to make complete revolutions. The axis must at least be able to bear four times the weight of the body. (41, 46).

12. A sphere swings about a horizontal tangent. Its pressure on the axis in its lowest position is less than the pressure which it would exert if its whole mass were collected at its centre of oscillation and oscillated by a weightless string by $\frac{2}{5}Mr\omega^2$, r being the radius of the sphere and ω its angular velocity in the position contemplated. (46).

13. Two equal rods can turn freely in their own plane about a hinge at the lower extremity, and have their upper extremities connected by a horizontal string. If the rods revolve about a vertical axis through the hinge and in their plane with an uniform angular velocity ω, the tension of the string

$$= \tfrac{1}{2}Mg \tan \alpha + \tfrac{1}{3}M\omega^2 a \sin \alpha,$$

where M is the mass and a the length of each rod, 2α the angle between the two.

14. A pendulum consists of a cylinder of one metal hung by a rod of another metal which passes through an orifice in direction of its axis and is attached to the centre of its base. Find the relation between the dimensions of the parts and expansibilities of the metals that the system may form a compensation pendulum.

15. Two spheres are connected by a thin rod whose ends penetrate each of them and are attached to them at their centres. Find the relation between the dimensions and expansibilities of the parts that this system may form a compensation pendulum when it oscillates about a given point of the rod.

16. The two parts m, m' of a compound vibrating body have their centres of gravity H, H', centres of oscillation O, O' and point of connexion P all in a straight line through S the point of suspension: $SH = h$, $SH' = h'$, $SO = l$, $SO' = l'$, $SP = e$, rr' the expansions of linear units of the masses for a given rise of temperature. The length L of the compound pendulum is unaltered by temperature if

$$\frac{L}{2} = \frac{mrhl + m'r'h'l' + m'h'e(r - r')}{mrh + m'r'h' + m'e(r - r')}.$$

17. A rod is supported at its two extremities and rests in a horizontal position. If one of the supports is removed so that the rod begins to turn about its other extremity, the pressure on the other support is suddenly diminished by one half. (46).

18. A rod is swinging about one end and when it is horizontal, the other end is suddenly fixed. The impulses on the points of attachment of the two ends are as 2 : 1. (59).

19. A square is revolving about one of its sides as a fixed axis with a given angular velocity. Find the blow which applied at one of its angular points perpendicularly to its plane will at once destroy its motion. (58). Determine also the accompanying impulse on the axis. (59).

20. A circle is revolving about a tangent line in its plane. If it be struck at any point except in the diameter perpendicular to that tangent, there must be a fracture at the point of attachment. (59).

21. An elastic rod swinging about its upper end impinges on a fixed vertical plane perpendicular to the plane of its motion. Its angular velocity is reversed and altered in the ratio of $1 : e$ by the impulse. (58).

22. A rod of mass M and length $2a$ hangs vertically downwards from one end. It will be thrown into its position of unstable equilibrium by the blow $= M\sqrt{3ag}$, the axis sustaining no impulse. (41, 58, 59).

23. A circle hanging at rest is capable of turning about a horizontal tangent line. Find how it is to be struck that there may be no impulse on the axis and that it may rise through 90^0. (41, 58, 59).

24. A rectangle capable of turning about one of its diagonals is struck perpendicularly at one of its angular points. Find the impulse on the axis and shew whether any form of the rectangle will allow it to vanish. (59).

25. When a lamina in the plane of xy capable of turning about the axis of y has a centre of percussion, the blow must be perpendicular to its plane and at a point whose co-ordinates are

$$x' = \frac{\Sigma(mx^2)}{\Sigma(mx)}, \qquad y' = \frac{\Sigma(mxy)}{\Sigma(mx)}. \quad (60).$$

26. A sector of a circle whose radius is a and angle α, is capable of turning about the axis of z which is in its plane and perpendicular to one of its bounding radii. If this radius be axis of x, and the centre the origin, the co-ordinates of the centre of percussion are

$$x' = \frac{3a}{8}\left(\frac{a}{\sin\alpha} + \cos\alpha\right), \qquad z' = \frac{3a}{8}\sin\alpha. \quad (60).$$

27. A circular right cone is capable of turning about an axis through its vertex. If it be struck perpendicularly to its curved surface, it does not admit a centre of percussion. (60).

28. A body capable of turning about a fixed axis is struck by a couple in the plane $Lx+My+Nz=0$, some point of the axis being origin and the principal axes at the same point the axes of co-ordinates. Prove that the greatest angular velocity will be produced if the axis be so taken that its equations are $\frac{Ax}{L}=\frac{By}{M}=\frac{Cz}{N}$, A, B, C being the principal moments of inertia. [Compare (102)].

III. *Motion of a rigid body about a fixed point.*

1. When a body is turning about the origin fixed the velocities of points of it which are in the axes of x, y, z, and each at a distance a from the origin are at a given instant v_1, v_2, v_3 respectively. Find the equations to the instantaneous axis at that instant. (63).

2. If ω_x, ω_y, ω_z be the angular velocities about the co-ordinate axes by which the motion of a body about the origin may be exhibited, find the locus of the particles whose velocity is $a\omega_x$. (63).

3. The locus of points in a body revolving about a fixed point which have at a proposed instant the same given velocity is a circular cylinder.

4. If the motion of a body about a fixed point be exhibited by angular velocities ω_x, ω_y, ω_z about the fixed co-ordinate axes, or by angular velocities ω_1, ω_2, ω_3 about rectangular axes fixed in itself,

$$\omega_x^2+\omega_y^2+\omega_z^2=\omega_1^2+\omega_2^2+\omega_3^2.$$

5. Find the nature of the motion in space of a body turning about a fixed point when its angular velocities about its principal axes are $\omega_1=a\sin nt$, $\omega_2=a\cos nt$, $\omega_3=n$ a con-

stant. (75). Shew that the instantaneous axis describes a circular cone in the body with uniform motion, and that the angular velocity about it is invariable.

6. If the body in (75) moves so that (1) the inclination of the planes AB and XY increases uniformly, and (2) the plane AC constantly passes through Z,

$$\omega_1 = -\frac{d\psi}{dt}.\sin(\alpha t + \beta), \quad \omega_2 = \alpha, \quad \omega_3 = \frac{d\psi}{dt}.\cos(\alpha t + \beta).$$

7. A rod can turn freely about its upper end. Find the nature of its motion when it revolves so as to retain a constant inclination to the vertical. Find also the pressure on the upper end.

8. If in the circumstances of (81),

$$\left(\frac{1}{C} - \frac{1}{B}\right)\tan^2\phi = \frac{1}{A} - \frac{1}{C}$$

at any instant when θ is finite, then ϕ will be invariable and ψ will increase uniformly. Find the values of ω_1, ω_2, ω_3 in this case.

9. If in the circumstances of (75) and (78), θ is invariable while ϕ and ψ increase uniformly, $L \propto \sin\phi$, $M \propto \cos\phi$ and $N \propto \sin 2\phi$. If the figure be one of revolution about the axis C, the forces acting on the body must be reducible to a force at the origin and a couple in the plane ZC.

If θ and ϕ increase uniformly while ψ is invariable, $L \propto \cos\phi$, $M \propto \sin\phi$, and $N \propto \sin 2\phi$. If the body be one of revolution about C, the forces must be reducible to a force at the origin and a couple in the plane through C perpendicular to ZC.

10. If in the notation of (78), $A = B$, $L = a\sin nt$, $M = a\cos nt$, $N = 0$, and if when $t = 0$ the instantaneous axis was coincident with the principal axis to which C belongs,

$$A\omega_1 = \frac{a}{m-n}(\cos nt - \cos mt),$$

$$A\omega_2 = \frac{a}{m-n}(\sin mt - \sin nt),$$

where $Am = (A - C)\omega_3$.

Examine the forms which ω_1, ω_3 assume when $m = n$.

11. A cube fixed at its centre of gravity and acted on by no force besides the reaction of the fixed point and gravity, will continue to revolve about any axis about which it was originally put in motion. (90). The pressure on the fixed point is equal to the weight of the body.

12. If a circular right cone whose altitude a is double the radius of its base turn about its centre of gravity fixed, and be originally put in motion about an axis inclined at an angle α to its axis of figure, the vertex of the cone will describe a circle whose radius is $\frac{3}{4}a \sin \alpha$. (89, 91).

13. If a right circular cone have a motion impressed upon it about a given axis through the centre of gravity whose equations when the centre of gravity is origin and the cone's axis the axis of z are $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, the invariable plane is

$$0 = \tfrac{1}{8}(a^2 + 4b^2)(lx + my) + b^2 nz$$

a being the altitude of the cone, b the radius of its base. (91).

14. With the notation and circumstances of (81) $\frac{k^2}{h^2}$ is intermediate to the greatest and least of the three quantities A, B, C.

15. A circular plate revolves about its centre of gravity fixed. If an angular velocity ω were originally impressed upon it about an axis making an angle α with its plane, a normal to the plane of the disc will make a revolution in space in time $\frac{2\pi}{\omega\sqrt{1 + 3\sin^2\alpha}}$. (86).

16. A solid of revolution moving about its centre of gravity has an angular velocity impressed upon it about a line between its axis of figure and that of z. If its axis of figure originally makes an angle α with the fixed axis of z and an angle β with the initial axis of rotation, and if

$$A \tan \beta = C \tan \alpha,$$

the axis will uniformly describe a circular right cone about the fixed axis of z, C, A being the body's moments of inertia about its own axis of figure and about a perpendicular line. (89, 91).

17. If a solid of revolution moving about its centre of gravity be originally put in motion about an axis with respect to which the moment of inertia is Q, and if C be the moment of inertia about the axis of figure, A that about a perpendicular axis through the centre of gravity, then

(1) The instantaneous axis will describe a circular cone in space, and the vertical angle of this cone is greatest when Q is a harmonic mean between A and C. (89, 91).

(2) The axis of figure will describe a circular cone in space whose semivertical angle is $\tan^{-1}\left\{\frac{A^2(Q-C)}{C^2(A-Q)}\right\}^{\frac{1}{2}}$.

(3) The instantaneous axis will describe a circular cone relatively to the axis of figure whose semivertical angle is

$$\tan^{-1}\left(\frac{Q-C}{A-Q}\right)^{\frac{1}{2}}. \quad (94).$$

18. When a body moves about a fixed point under no force besides the reaction of that point, shew that no point of it will in general describe areas uniformly on the co-ordinate planes.

19. A cube fixed at its centre of gravity is struck in direction of an edge: find the axis about which it begins to rotate. (99).

20. A cube fixed at its centre of gravity is struck by three equal blows in directions of three of its edges which neither meet nor are parallel. Determine the axis about which it begins to revolve. (99).

21. A rod capable of turning about its centre of gravity is so set in motion as to describe a given right cone. Find the nature of the impulsive forces which originally caused its motion. (91, 99).

22. A cone is moving about its vertex. Shew that it cannot be struck perpendicularly to its surface so that its motion may be destroyed without producing an impulse at the vertex.

23. If a rigid body capable of turning about a fixed point be struck by impulses, the sums of whose components in directions of the co-ordinate axes are X, Y, Z, and if F, G, H be the consequent impulses in similar directions on the fixed point, $\bar{x}$, $\bar{y}$, $\bar{z}$ co-ordinates of the centre of gravity, the axes being principal axes,

$$0 = (X - F)\bar{x} + (Y - G)\bar{y} + (Z - H)\bar{z}. \quad (101).$$

If ω_1, ω_2, ω_3 be the initial angular velocities about the co-ordinate axes,

$$0 = (X - F)\omega_1 + (Y - G)\omega_2 + (Z - H)\omega_3.$$

24. A triangle ABC fixed at its centre of gravity is struck perpendicularly to its plane at its right angle C. The initial instantaneous axis is parallel to the hypothenuse. (99).

25. The motion of a body revolving about the origin fixed is exhibited by the angular velocities ω_x, ω_y, ω_z about the rectangular cordinate axes. If the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ suddenly becomes a fixed axis, obtain equations for determining the impulses on this axis and the new angular velocity about it.

26. An elliptic lamina fixed at its centre is struck and begins to rotate about the diameter $ay - bx = 0$. The blow must have been applied at some point of the diameter

$$ay + bx = 0,$$

the axes of the figure being axes of co-ordinates. (99, 102).

Will the body continue to rotate about this diameter if no force except the reaction of the fixed point henceforth acts upon it? (88). Will the angular velocity about its instantaneous axis continue invariable?

27. An ellipsoid whose centre is fixed receives a normal blow so that the initial instantaneous axis lies in the conical surface

$$\frac{x^2}{b^2 - c^2} + \frac{y^2}{c^2 - a^2} + \frac{z^2}{a^2 - b^2} = 0.$$

The blow must have been given at some point where the ellipsoid meets the conical surface

$$\frac{b^2 - c^2}{(b^2 + c^2)^2} \cdot \frac{a^4}{x^2} + \frac{c^2 - a^2}{(c^2 + a^2)^2} \cdot \frac{b^4}{y^2} + \frac{a^2 - b^2}{(a^2 + b^2)^2} \cdot \frac{c^4}{z^2} = 0. \quad (99).$$

28. A solid ellipsoid whose centre is fixed receives a normal blow at a point α, β, γ of its surface: the equations to the initial instantaneous axis when the axes of the body are co-ordinate axes are

$$\frac{x\alpha}{a^2} \cdot \frac{b^2 + c^2}{b^2 - c^2} = \frac{y\beta}{b^2} \cdot \frac{c^2 + a^2}{c^2 - a^2} = \frac{z\gamma}{c^2} \cdot \frac{a^2 + b^2}{a^2 - b^2}. \quad (99).$$

If the ellipsoid thus put in motion moves henceforth under no force but the reaction of its fixed centre, the equation to its invariable plane when the original directions of its axes are co-ordinate axes is

$$0 = a^2(b^2 - c^2)\frac{x}{\alpha} + b^2(c^2 - a^2)\frac{y}{\beta} + c^2(a^2 - b^2)\frac{z}{\gamma}. \quad (91).$$

IV. *Motion of a free rigid body.*

1. If the velocities of two points of a rigid body be given in magnitude and direction, shew that the translation and rotation which exhibits its motion can in general be determined. Shew in what cases these data are insufficient.

2. A body has a motion of rotation ω about the axis $\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, where $l^2+m^2+n^2=1$. The motion is equivalent to rotations $l\omega$, $m\omega$, $n\omega$ about the co-ordinate axes, and translations $(m\gamma - n\beta)\omega$, $(n\alpha - l\gamma)\omega$, $(l\beta - m\alpha)\omega$ in direction of them. (106).

3. A circular disc revolves with an uniform angular velocity ω about an axis through its centre perpendicular to its plane, while its centre describes a circle of radius a with another uniform angular velocity Ω about a point in the plane of the disc. The motion is at any instant exhibited by a single rotation, and the locus of the instantaneous axis is a cylinder of radius $\frac{a\omega}{\omega+\Omega}$. Explain the result when ω, Ω are equal and contrary in direction.

4. The motion of the earth will not generally be exhibited by rotation about a single axis, neglecting the nutation of its axis and supposing its centre to describe an ellipse. (95, 115).

5. The motion of a rigid body is exhibited by the coexistence of given rotations about a series of given parallel axes; determine the single rotation to which the motion is in general reducible.

6. A body has equal angular velocities about two axes which neither meet nor are parallel. Prove that the central axis of the motion is equally inclined to each of the axes. (112).

7. A given weight hangs by a string which is wrapped round a wheel whose plane is vertical. Compare the motions produced in the wheel when its centre is fixed and when it is allowed to fall freely, the system being originally at rest. (118).

8. A wheel revolves about an axis through its centre perpendicular to its plane with a velocity increasing uniformly from zero, while its centre moves with an uniform velocity along a straight line in the plane of the wheel. The instantaneous axis has a hyperbolic cylinder for its locus, and the forces producing such a motion must form a couple of constant moment.

9. If a body be under no forces in motion about a principal axis through the centre of gravity, that axis will continue fixed in space and will always be the axis of instantaneous rotation about which the body turns with uniform angular velocity. (88, 118).

10. A hemisphere is acted upon by a force constant in magnitude and direction applied at the centre of its base. If the radius in which its centre lies is originally inclined at a small angle to the direction of the force, the body will make small oscillations about this direction in the time $\pi \sqrt{\frac{83}{120} \cdot \frac{a}{f}}$ while the centre of gravity moves in the same direction, a being the radius of the hemisphere and f the given accelerating force. (118). In what unit of time will this result be expressed?

11. How must a free cube be struck that it may begin to rotate (1) about a diagonal, or (2) about an edge? (125).

12. A rectangular lamina whose sides are a, b is struck perpendicularly at a point whose co-ordinates from the fixed centre parallel to the sides of the rectangle are h, k. The rectangle will begin to revolve about the line $0 = \frac{b^2 x}{k} + \frac{a^2 y}{h}$.

If the body were free, it would begin to revolve about a line in itself parallel to the former and at the distance $\frac{1}{12}\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)^{-\frac{1}{2}}$ from it. (100, 125).

13. A free spheroid is put in motion so that when it is thenceforth acted on by no force, its centre moves along a given straight line while its axis describes a circular cone about the same line. Find the nature of the impulsive forces which produced its motion. (89, 124).

14. A circular lamina of radius a, revolving about a diameter with a given angular velocity ω, is struck perpendicularly to its plane at the extremity of the diameter at right angles to the former, and afterwards has an angular velocity ω'. The distance between the original and latter axis of rotation $= \frac{a}{4}\cdot\frac{\omega'-\omega}{\omega'}$. (124).

15. Find at what point a rod AB may be struck perpendicularly that one end of it A may be initially at rest. (124). Shew that the point of impact is the centre of percussion when the rod can turn about A fixed. If the rod move henceforth freely, the path of A is given by

$$x = a\cos^{-1}\frac{y}{a} - \sqrt{a^2-y^2},$$

where the axis of x is measured from the original position of the centre of gravity in direction of the blow, and $2a$ is the length of the rod. (123).

16. A cube is struck by three equal blows in directions of three of its edges which neither meet nor are parallel. Determine its initial instantaneous axis. (124, 126).

V. *Miscellaneous examples.*

1. A circular disc of radius a rolls with uniform velocity V along a horizontal plane: the effective force on any particle of it at distance r from the centre is $\frac{V^2}{a^2} r$ acting in the line joining the particle with the centre of the disc. (132).

2. A sphere rolls on a rough board which itself can slide horizontally on a fixed smooth horizontal plane on which it lies, the motions of the sphere and board being in one vertical plane. Determine the motion. (132).

3. A sphere can turn about its centre which is fixed. Another sphere is placed upon it at a given point and rolls down it. Determine the motion. (132).

4. A sphere rolls on a horizontal plane which is itself made to move horizontally with an uniformly increasing velocity. Determine the friction between the sphere and plane. (132). If the plane is suddenly brought to rest, find the change in the velocity of the sphere.

5. A cylinder rolls down a plane inclined at 30^0 to the horizon. If a be the radius, l the space through which it has descended, x, y the co-ordinates of a point of the cylinder referred to axes parallel and perpendicular to the plane originating in the line of contact, then the effective forces on that particle in these directions are

$$\frac{gy}{3a} + \frac{2glx}{3a^2} \text{ and } \frac{gx}{3a} + \frac{2gl}{3a^2}(a - y).$$

6. A rough sphere rolls on a horizontal plane under the action of a centre of force in the plane. The equation to the projection on the plane of the path of the sphere's centre is $\frac{d^2u}{d\theta^2} + u = \frac{5}{7} \cdot \frac{P}{h^2u^2}$, where P is the resultant force resolved parallel to the plane.

7. If a wheel rolls on a plane the velocity of translation of its centre is equal to the product of its angular velocity about its centre by its radius. (132).

8. Given the motions of each of a system of unconnected particles, find the impulse to be applied to each that the centre of gravity of the system may afterwards be at rest.

9. A bomb shell projected in a given manner explodes in its flight. If a fragment of given mass impinges on the ground with a given velocity and is suddenly brought to rest find the change in the direction of the parabolic path which the centre of gravity of the whole has described. (139, 140).

10. Two equal particles are describing an ellipse in the same direction about the same centre of force at their centre of gravity. If they suddenly become rigidly united they will then revolve with the uniform angular velocity $\frac{ab\sqrt{\mu}}{r^2}$, where μ is the absolute force, a, b the semi-axes of the ellipse, and $2r$ the distance of the particles when the change takes place. (142, 143).

11. If a small planet were to explode and if the fragments then describe ellipses about the sun's centre, a, b being the semi-axes of the ellipse described by a fragment of mass m in a plane inclined to the ecliptic at an angle i, $\Sigma\left(m\sqrt{\frac{b^2}{a}}\cos i\right)$ is independent of the place where the explosion takes place or the manner in which the body is separated by it, the summation indicated being made through all the portions of the original body. (142, 143).

12. A cylinder, capable of turning about its horizontal axis, has a cylindrical shell enclosing it and fitting closely to it so that there is friction between them. Given the different angular velocities with which each is at a given time revolving about the axis, find the common angular velocity which they ultimately attain. (142).

13. A series of concentric spherical shells fit closely one within another and influence one another's motion by friction. If the shells have given rotations impressed upon them about given diameters, no extraneous force acting on the system, they will ultimately revolve about a common axis which is perpendicular to the plane

$$0 = x\Sigma(Al\omega) + y\Sigma(Am\omega) + z\Sigma(An\omega);$$

where A is the moment of inertia about a diameter of a shell which originally has an angular velocity ω about the diameter $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, the centre of the shells being origin. (90, 144).

14. A sphere whose radius is a and mass M has an angular velocity ω about its diameter which is perpendicular to the plane of xy while its centre is describing a circle about the origin in the plane of xy with an angular velocity Ω. For this body the value of the function $\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$ is $\frac{2}{5}Ma^2\omega + Mb^2\Omega$, (150).

[This example suggests an oversight in Laplace's determination of the invariable plane of the solar system. Mech. Celeste. Liv. VI. ch. 17.*]

15. A wheel rolls on a horizontal plane. The sum of its vires vivæ from the translation of its centre of gravity and its rotation about its instantaneous axis exceeds the vis viva of the body by Mv^2, M being the mass of the body and v the velocity of its centre. (154).

16. A sphere revolving about a diameter and acted on by no extraneous force expands symmetrically. Its vis viva is proportional to its moment of inertia about a diameter inversely. (142, 153).

17. A wheel rolls on a horizontal plane. Prove that its velocity is uniform.

* Poinsot. Elemens de Statique. Appendix. 1842.

18. Find the time in which a sphere will roll down a given inclined plane, and the amount of friction at any instant of the motion.

19. A solid cylinder fits exactly within a hollow cylinder so that their axes coincide and no friction can take place between them. The whole rolls down a rough inclined plane. Compare the time in which the system descends through a given space with that in which it would descend through the same space if the two cylinders were rigidly united. (154, 156).

20. A rod of length $2a$ descends in a vertical plane, its lower end sliding on a smooth horizontal plane. If the rod is originally inclined at 30^0 to the horizon, its angular velocity when it becomes horizontal is $\sqrt{\frac{3g}{4a}}$.

21. When a sphere rolls on a fixed plane under the action of any forces through its centre, the instantaneous axis is always perpendicular to the direction of motion of the centre of gravity, and the friction is opposite to the direction of the resultant force parallel to the plane.

22. A sphere rolls on an inclined plane. The path of its centre of gravity is in general a parabola.

23. Determine the motion of a sphere rolling down a smooth inverted cycloid whose axis is vertical. (156).

24. Shew that the principle of vis viva is not true in the following cases.

A sphere slides along a horizontal plane which rises with a velocity proportional to the fourth power of the time.

Two particles are connected by a string which increases uniformly in length. One of them hangs over the edge of a table and moves the other which lies on the table.

A sphere expands while it falls freely under the action of gravity.

A sphere rolling down an inclined plane contracts in size symmetrically, so that $a - \phi(t)$ is its radius at time t from the beginning of the motion.

Obs. In each of these cases it will be instructive to ascertain the equation of the kind $L = 0$, involved in the proof of the principle of vis viva, which does not fulfil the conditions on which the proof rests.

25. A carriage with four wheels descends an inclined plane. Find the velocity at the bottom. Compare this with the velocity which would be acquired (1) if two of the wheels be prevented from turning, (2) if the plane were smooth. (156).

26. Two wheels work together by means of a band, so that a weight hanging from one raises another weight hanging from the other. Determine the motion (156).

27. A machine wherein the efficiency exhibited is to that applied as $\alpha : 1$, raises a volume V of water through a height h, the efficiency applied to it $= \frac{\sigma V h}{\alpha}$, σ being the specific gravity of water (162).

An efficiency 33000, a foot and a pound being the units, exerted in a minute of time is termed a horse power. Hence if the water in the preceding question be raised in t minutes, the agent in operation exerts $\frac{\sigma V h}{33000\, \alpha t}$ horse powers, σV and h being expressed with reference to a pound and a foot.

28. A globe of given exterior radius rolls down an inclined plane d feet long in n seconds. Shew how to ascertain from these data whether the globe is hollow or solid; and if the former be the case, shew how the interior radius may be known, the density being uniform.

29. Two equal rods united at their ends at a given angle rest over a fixed horizontal cylinder at stable equilibrium in a vertical plane. Find the time of a small oscillation when the system is slightly displaced in the vertical plane through the rods.

30. A rod of length a is placed in a smooth hemisphere whose radius is a, so that a plane through the rod and the centre of the hemisphere is always vertical. The rod will oscillate in the same time as a pendulum whose length is $\frac{5}{9}\sqrt{3}\,.\,a$.

31. A horizontal rod of length $2a$ hangs by two parallel strings each of length b attached to its ends. If it be slightly displaced in direction of its length it will make small oscillations in the time $\pi\sqrt{\frac{b}{g}}$. If it be twisted horizontally through a small angle, it will make small oscillations in the time $\pi\sqrt{\frac{b}{3g}}$. (156).

32. A smooth ball is placed within a spherical surface and lies at rest in it. If the spherical surface be made to move horizontally so as to have at any time the velocity which gravity would generate in a falling body, the centre of the ball will rise into the same horizontal plane with the centre of the spherical surface.

33. If a figure of revolution resting at stable equilibrium on its vertex on a horizontal plane be slightly disturbed, it will make small oscillations in the time $\pi\sqrt{\frac{k^2}{c}}$, if the plane is smooth, or $\pi\sqrt{\frac{k^2+c^2}{c}}$ if it is rough, where c is the distance of the centre of gravity from the centre of curvature of the generating curve at the vertex. Are the results altered if the plane has a given uniform horizontal motion?

34. The half of a circular cylinder cut by a plane along its axis rocks on a smooth horizontal table. The instantaneous axis lies always on the surface of a circular cylinder whose radius is to that of the body as 3 : 8.

35. A wire bent into the form of a circular arc makes small oscillations in a vertical plane between two smooth inclined planes. The length of the simple equivalent pendulum is $\frac{\text{radius} \times \text{arc}}{\text{chord}}$.

36. Two equal weights connected by a weightless rod of length $a\sqrt{3}$ are placed in a smooth hemisphere whose radius is a, so that a plane through the rod and the centre of the hemisphere is vertical; the rod if disturbed from the position of equilibrium in the same plane will make oscillations in the same time as a simple pendulum of length $2a$.

37. A rough hemisphere rests at stable equilibrium on a fixed rough sphere. If the former be slightly displaced determine the subsequent motion (1) when its flat base (2) when its curved surface is in contact with the fixed sphere.

38. A rough plank capable of turning about a horizontal axis through its centre of gravity is in a horizontal position. Determine the motion of a rough sphere placed upon it at a given point.

39. Determine the motion of a rod which passes through a small fixed ring while its lower end slides on a smooth horizontal plane.

40. A wheel is set in motion on a rough horizontal plane with given velocities of translation and rotation. Find the locus of the instantaneous axis until friction reduces the motion to that of rolling.

41. A heavy right angled cone resting on a horizontal plane with its vertex downwards is bisected by a vertical plane through its axis. The relative instantaneous decrease of pressure on the plane $= \frac{4}{3\pi^2}$, and the initial mutual pressure at the vertex $= \frac{1}{2\pi}$. weight of cone.

42. Three equal spheres lie on a table in contact, their centres forming an equilateral triangle. A fourth equal sphere

drops and impinges on the three similarly and simultaneously. The impinging ball will entirely lose its velocity if $e = \frac{1}{6}$.

If the three quiescent balls had a string passing round them in the plane of their centres, find the impulsive tension which the string sustains when the impact takes place.

43. Two equal smooth balls moving in parallel straight lines with a common velocity V and impinge at once on a third equal ball lying at rest, so that the centres of the three at the moment of impact form an equilateral triangle in the plane of the motion of the two balls, to one of the two sides of which the direction of V is perpendicular. The latter ball will begin to move with the velocity $\frac{3}{5}(1 + e)\, V$.

44. Two equal spheres are in contact: find how one of them should be struck that after impact it may move in a given direction.

45. A sphere rolling on a plane impinges on a perfectly rough point. Find the condition that it may be brought to rest.

46. Two rods are moving in the same plane with given velocities of translation and rotation. The end of one rod strikes the centre of gravity of the other: determine the subsequent motion.

47. A cylinder sliding in a direction perpendicular to its length along a smooth plane suddenly comes to a rough part of the plane whereby it is compelled to roll. Prove that it loses at once one third of its velocity.

48. A hoop is rolling on a horizontal plane. If it be struck at a given point of its circumference, find the limitations to the direction of the blow that the motion may continue that of rolling.

49. A rod slides between a smooth horizontal and vertical plane until a string connecting its lower end with the intersection of those planes becomes tight, the string and the motion of the rod being in one vertical plane. The rod may be brought to rest by the impulse if $\tan^2\alpha < 2$, α being the rod's inclination to the horizon when the impulse takes place.

50. A rod descends in a vertical plane, its lower end sliding on a smooth horizontal plane, and impinges on a given fixed peg. Determine its motion immediately after the impulse.

51. An inelastic rod rests in a horizontal position on two pegs with its centre of gravity equidistant from them. If it be turned about one of them and then allowed to fall, sliding being prevented by friction, its motion will cease or not after the first impact as the distance between the pegs is greater or less than $\frac{1}{\sqrt{3}}$ × the length of the rod.

52. A circular disc revolving in its own plane drops on a rough inelastic inclined plane whose intersection with the horizon is perpendicular to the plane of the disc. The disc will run up or down the plane according as $\frac{1}{2}a\omega >$ or $< V\cos i$, V and ω being the velocities of translation and rotation of the disc before impact, a its radius and i the inclination of the plane to the vertical.

53. A chain coiled loosely on a horizontal table is drawn up by a very thin string at one end of it, passing over a pulley and having a given weight at the other end. The pulley is vertically above the coil of chain and the size of the coil may be neglected so that each link begins to rise vertically. Determine the velocity of the chain when it reaches the pulley.

Determine the motion also if the chain originally lies in a horizontal line of which one end is vertically under the pulley.

54. A rod is constrained to move vertically by two fixed rings through which it slides, while its lower end presses upon

an inclined plane sliding on a horizontal table. Find the velocity generated in the inclined plane when the rod leaves it and the pressure on each ring at any instant. If at a given instant the plane be suddenly brought to rest by a fixed obstacle, determine the impulse on each ring.

55. Two equal rods connected by a hinge are placed on two pegs in the same horizontal line, the rods having the same inclination to the vertical. Determine the angle through which they open.

If at a given point of the motion a string connecting the lower ends of the rods becomes tight, find the impulse on each peg.

56. Three equal rods are placed at given equal inclinations to the horizon and to one another, united by a common joint at their lowest point. A sphere is placed between them. Determine its motion. If the upper ends of the rods be united two and two by equal strings which are originally slack, determine the impulsive tension which each sustains when they become tight.

57. An inelastic and smooth ellipsoid whose semi-axes are a, b, c having a velocity of rotation ω about one axis c impinges with a velocity v on a sphere of equal mass. At the instant of impact the sphere touches the ellipsoid at the extremity of the latus rectum of its principal section containing the axes a and b, and the axis a is in direction of the motion of the centre of gravity. If the eccentricity of that principal section is $\sqrt{\frac{2}{5}}$, and if the ellipsoid has no rotation after impact, $2a\omega = v$.

58. A circular hoop revolves about a vertical diameter with uniform angular velocity, and a smooth heavy ring slides upon it. Find the condition that the ring may perform oscillations about the lowest point.

59. Two equal rods are similarly placed in one vertical plane so that their lower ends are on a smooth horizontal table and their upper ends press against one another. Find

whether the rods will cease to be in contact before they fall to the table.

60. Two particles connected by an elastic string move with equal velocities in the same straight line, their distance being the length of the string when it sustains no tension. If the hinder of the two be suddenly stopped, find how far the other body will move.

THE END.

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